Choice Dominance and Single Crossing Indifference Curves: a Revealed Preference Analysis

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Choice dominance and single crossing indifference curves: a revealed preference analysis

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Abstract

We introduce a behavioural condition, called *choice dominance*, which (partially) ranks individuals based on their consumption of a certain good. The notion is equivalent to a single crossing restriction on the indifference curves of the individuals. We provide a revealed preference condition, called X-GARP, to test for choice dominance and we incorporate the notion into a household framework to obtain testable restrictions on the change in household demand after an increase in the reservation utility for one of the household members. We apply our findings to an experimental dataset and a budget survey from a conditional cash transfer programme.

Keywords: Choice dominance, Single crossing indifference curves, Revealed preferences, Conditional cash transfers
JEL codes: C14, C60, D11, D12

Different people have different tastes: what you prefer might be different from the things someone else prefers. But what does it mean that you prefer something more than someone else? And how can we verify that this is indeed the case? These are the questions we address in this paper.

We look at a setting with two individuals, Ann and Bob, and two goods, x and y. We consider the case where Bob has a stronger preference for x compared to Ann and formalise this notion by means of a behavioural condition which we call *choice dominance*. It captures

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the idea that –regardless of the price configuration– Bob always finds it optimal to consume at least as much of x compared to Ann (and Ann consumes more of y compared to Bob).

A first result links this behavioural condition to a Spence-Mirrlees type of single crossing condition on the indifference curves of Ann and Bob (Mirrlees, 1971; Spence, 1973). Intuitively, it arises when Bob's indifference curves are steeper compared to Ann's when plotted in the x-y plane. Although single crossing conditions are prevalent in economics, our framework only imposes minimal conditions on the underlying utility functions as we abstain from imposing any smoothness or strict monotonicity assumptions. This means that utility functions, although quasi-concave, are not required to be differentiable.

Our main contribution presents a revealed preference characterisation for choice dominance. Revealed preference analysis was pioneered by Samuelson (1948) and Houthakker (1950) and further explored in the seminal contributions of Afriat (1967), Varian (1982) and Diewert and Parkan (1985). Attractively, it does not impose a parametric functional specification on the utility functions and is therefore robust to such specification errors.

In practical terms, we consider a finite consumption data set for Ann and Bob in terms of prices and chosen quantities for x and y. We then say that such data is rationalizable by choice dominance if there are rational demand correspondences, one for Ann and one for Bob, that explain their observed choices and such that Bob choice dominates Ann in terms of good x. In addition to the standard rationality conditions, we derive an intuitive necessary and sufficient condition for choice dominance which we call X-GARP. It requires the absence of a GARP (Generalized Axiom of Revealed Preference) violation in the pooled observed consumption choices in cases where Bob faces a budget with a lower relative price for good x compared to Ann. The X-GARP condition is intuitive, easy to verify and provides a first revealed preference test for comparing preferences between individuals.

We illustrate our X-GARP test using an experimental dataset from Porter and Adams (2015). In that paper subjects play a series of dictator games with either their parents or strangers as recipients. Subjects decide on the amount to keep (good x) or give (good y) under different price regimes. Referring to P as the situation in which the recipient is a parent and S to the case where the recipient is a stranger, we expect that the S choices should dominate the P choices for keeping (good x), as this embeds the idea that subjects are more generous towards their parents compared to strangers. Our findings confirm this hypothesis.

Next, we show how our notion of choice dominance and its RP characterisation can be fruitfully applied to test hypotheses on responses of household behaviour to policies. To that end, we first theoretically show how one can include the notion of choice dominance to a collective household setting. In particular, we focus on a situation where two spouses Ann and Bob live together in a household and jointly decide on the consumption of a bundle (x, y). Following the literature on collective models, we assume that the household makes Pareto optimal decisions (Browning and Chiappori, 1998; Chiappori and Mazzocco, 2017). We demonstrate that if Bob's reservation utility (i.e., bargaining power) increases and if Bob choice dominates Ann for good x, then the household demand will change in favour

of bundles that contain more of x. We use this result to derive an X-GARP restriction on the observable household choices after an unambiguous change in of one of the household members' reservation utility.

We then show how we can apply this extension of choice dominance to a collective setting to household survey data. In particular, we use survey data pertaining to PROGRESA, which is a conditional cash transfer programme launched in rural Mexico in the late 1990s. The programme provided cash payments to families conditional on their children attending school. The cash payments were also targeted towards the mother in the family. We focus on a sample of eligible households (based on household characteristics) and exploit the fact that the roll out of the programme during the initial phase was randomised across localities included in the evaluation sample, allowing us to define 'treatment' and 'control' villages. This randomisation also creates exogenous variation in the share of household income controlled by the mother, which has been used earlier to test for the efficiency assumption in the collective household model (Attanasio and Lechene, 2014; De Rock et al., 2022).[1]

In our setup, being in a treatment village implies that mothers have access to the receipt of the cash transfers, which plausibly increased their reservation utility. We test the assumption that mothers have a stronger preference than fathers for highly nutritious food, such as vegetables, fruits, and proteins. Our results support this hypothesis, in line with previous studies on the influence of cash transfers on dietary choices (Hoddinott, Skoufias and Washburn, 2000; Hoddinott and Skoufias, 2004; Angelucci and Garlick, 2015).

The two goods-two person setting Though somewhat restrictive, the two person two goods setting still allows us to capture many relevant features.

The focus on two individuals can easily be relaxed: by ranking any two individuals on the basis of their preferences, we can partially rank any set of individuals, by transitivity. As such it is straightforward to extend our analysis to more than two individuals.

The two-goods setting is more difficult to relax. Fortunately, a multi-goods setting can often be reduced to the two-goods case. A first setting is through the use of the Hicksian aggregates: a set of goods can be represented as a Hicksian aggregate if the goods' relative prices remain fixed over observations (Hicks, 1936). It thus suffices to verify the empirical validity of constant relative prices, to check whether the demand for multiple goods can be reframed in terms of two Hicksian aggregates.[2] An alternative setting that validates the focus on two goods arises if preferences are weakly separable. If x_i is the quantity of good i and u is the consumer's utility function, we have that x_1 and x_2 are weakly separable from the other goods if the utility function can be written as $u\left(s\left(x_1, x_2\right), x_3, x_4, x_5, ...\right)$ where s is some subutility function for goods x_1 and x_2. In this case a two-stage budgeting

[1] These papers base their test on an estimated parametric demand system of different food categories and test whether the system satisfies the restrictions imposed by the collective household presented in Bourguignon et al. (2009).

[2] More general conditions on relative price movements to achieve a similar type of aggregation result are discussed in Lewbel (1996).

argument shows that the demands for x_1 and x_2 only depend on their relative prices and the total expenditure allocated to these goods.[3]

Related literature Single crossing conditions take a prominent role in several areas of economics. Foremost, as in our paper, they are used to obtain meaningful comparative static results.[4] Further, single crossing conditions are also used to obtain equilibrium existence conditions (Athey, 2001), and are often helpful for identification purposes (Apesteguia, Ballester and Lu, 2017). While single crossing assumptions are prevalent in economics they are rarely tested in practice. Our paper is one of the first that gives empirical testable restrictions on choice data resulting from such type of single crossing restriction.

The paper also connects to the revealed preference literature. In this respect, Chambers, Echenique and Shmaya (2011) look at the revealed preference conditions for the property that a pair of goods are gross substitutes. This means that if the price of one good increases then the demand for the other good increases. The notion focusses on the comparative statics between goods for the same individual. Our focus, instead is on a single crossing condition relating preferences of two distinct individuals. Next, Lazatti, Quah and Shirai (2024) look at a test for pure strategy Nash equilibrium play in games with monotone best response, and illustrate this using an entry game. They use a single-crossing condition on the profit functions of the firms to obtain monotone comparative statics.

Finally, a working paper of Heufer, van Bruggen and Yang (2022) looks at rationalizability by changing preferences for reciprocity in an experimental context. They study an axiom which they call *Agreement* that relates two different preferences, one more generous and another more selfish. To be precise, let a be an allocation of a monetary amount in which an individual gets less compared to an allocation b. Then, their Agreement axiom stipulates that if an individual prefers a over b according to her selfish preference relation, then she will also prefer a over b according to her generous preferences. The Agreement axiom has a clear single crossing flavour and is therefore close to our choice dominance condition. Their motivation, analysis and proof techniques, however, differ from ours.

Outline Section 1 introduces the basic notation and defines the concept of choice dominance. Section 2 connects choice dominance to a single crossing condition on the indifference curves. Section 3 provides the main revealed preference characterisation. Section 4 illustrates our revealed preference test on the experimental data from Porter and Adams (2015). Section 5 introduces choice dominance in a household model and derives the testable restrictions. Section 6 applies these in our second empirical application. Section 7 contains a conclusion. All proofs are in the Appendix.

[3]It is in fact possible to empirically check this weak separability structure. See, for example, Afriat (1969); Varian (1983); Diewert and Parkan (1985); Quah (2014) and Cherchye, Demuynck, De Rock and Hjertstrand (2015) for revealed preference conditions.

[4]See for example Milgrom and Shannon (1994) for a seminal contribution.

1 Choice dominance

Given x and $y \in \mathbb{R}$ we denote by $x \vee y$ their maximum and by $x \wedge y$ their minimum.[5] For two subsets $X, Y \subseteq \mathbb{R}$ we define,

$$X \vee Y = \{x \vee y | x \in X, y \in Y\} \qquad \text{and} \qquad X \wedge Y = \{x \wedge y | x \in X, y \in Y\}.$$

For any two subsets $Z, W \subseteq \mathbb{R}^2$ we define,

$$Z \curlyvee W = \{(x_Z \vee x_W, y_Z \wedge y_W) | (x_Z, y_Z) \in Z, (x_W, y_W) \in W\},$$
$$Z \curlywedge W = \{(x_Z \wedge x_Z, y_W \vee y_W) | (x_Z, y_Z) \in Z, (x_W, y_W) \in W\}.$$

We consider a consumption setting with two goods x and y and two individuals A and B. We will use the notation C or sometimes D to make a generic reference to either A or B.

A budget (p_x, p_y, m) specifies strictly positive prices $p_x, p_y > 0$ for goods x and y, and an income $m > 0$. The budget set equals the set of all feasible bundles $\{(x, y) \in \mathbb{R}^2_+ | p_x x + p_y y \leq m\}$. Without loss of generality we normalize the price of good y to one. Defining the relative price $\omega = p_x/p_y$ and normalized income $z = m/p_y$ we can summarize a budget by the pair $(\omega, z) \in \mathbb{R}^2_{++}$ and the resulting budget set by $\{(x, y) \in \mathbb{R}^2_+ | \omega x + y \leq z\}$.

We assume that A and B have monotone utility functions u^A and u^B.[6] Let $\mathcal{D}^C$ be the demand correspondence for individual $C \in \{A, B\}$:

$$\mathcal{D}^C(\omega, z) = \underset{x,y \geq 0}{\operatorname{argmax}} u^C(x, y) \text{ s.t. } \omega x + y \leq z,$$

We assume that for all budgets $(\omega, z) \in \mathbb{R}^2_{++}$, the demand correspondences $\mathcal{D}^A(\omega, z)$ and $\mathcal{D}^B(\omega, z)$ are non-empty. We now introduce our behavioural condition that captures the notion that individual B always chooses as least as much of x compared to individual A. As choice correspondences might be multi-valued we will need to take a stance on how we formalize the notion of 'at least as much'.

Definition 1 (Choice Dominance)**.** *Let $\mathcal{D}^A$ be the demand correspondence for A and let $\mathcal{D}^B$ be the demand correspondences for consumer B. We say that B choice dominates A in terms of x if for all budgets $(\omega, z) \in \mathbb{R}^2_{++}$,*

$$\mathcal{D}^B(\omega, z) = \mathcal{D}^A(\omega, z) \curlyvee \mathcal{D}^B(\omega, z) \tag{1}$$

and,

$$\mathcal{D}^A(\omega, z) = \mathcal{D}^A(\omega, z) \curlywedge \mathcal{D}^B(\omega, z). \tag{2}$$

We also write this as $B \succeq_x A$.

[5] So if $x \geq y$ then $x \vee y = x$ and $x \wedge y = y$.

[6] The function u^C is monotone if $(x, y) \geq (x', y')$ implies $u^C(x, y) \geq u^C(x', y')$ and $(x, y) \gg (x', y')$ implies $u^C(x, y) > u^C(x', y')$, where we say that $(x, y) \geq (x', y')$ iff $x \geq x'$ and $y \geq y'$, we state $(x, y) > (x', y')$ iff $(x, y) \geq (x', y')$ and $(x, y) \neq (x', y')$ and we write $(x, y) \gg (x', y')$ iff $x > x'$ and $y > y'$.

Conditions (1) and (2) imply that for any budget (ω, z) and any optimal choice of A, B will always have an optimal bundle that contains at least as much of x, compared to the optimal bundle of A. Likewise, for any optimal bundle of B, A has an optimal bundle containing no more of x compared to the optimal bundle of B.

Some remarks are in order. First, as the setting is entirely symmetric, we have that $B \succeq_x A$ if and only if $A \succeq_y B$. Second, if the demand correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$ are convex valued (which happens if, for example, the utility functions are quasi-concave), then the two conditions (1) and (2) are equivalent. Finally remark that, as the choice correspondences may overlap, choice dominance does not necessarily require that every optimal bundle of B contains more of x compared to any optimal bundle of A. In this respect, our notion of choice dominance seems to be most consistent with the non-uniqueness of demand which we allow for in our framework.

2 Single crossing indifference curves

Having introduced the notion of choice dominance one might wonder what type of preferences lead such choice behaviour. In this section, we relate choice dominance to a Spence-Mirlees type of single crossing condition on the indifference curves of the individuals.

To that end we will assume in this section, that the utility functions u^A and u^B are quasi-concave, continuous and monotone on $\mathbb{R}^2$.[7] Note that we abstain from imposing any smoothness conditions, in line with the usual revealed preference approach that generally avoids imposing any differentiability conditions.

Let us denote by $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ the extended set of non-negative real numbers. To define the single crossing condition, we need some additional notation referring to the slopes of the indifference curves. For a bundle $(x, y) \in \mathbb{R}^2_+$, we will denote by $\Omega^C(x, y) \subseteq \overline{\mathbb{R}}_+$ the set of all slopes of all supporting hyperplanes tangent to the indifference curve of individual C through the bundle (x, y). As the upper contour sets are convex and closed (from continuity and quasi-concavity) $\Omega^C(x, y)$ is a non-empty, closed and convex set.[8] As utility functions are monotone we have that $\Omega^C(x, y) \subseteq \overline{\mathbb{R}}_+$. Definition 2 formally defines these sets.

Definition 2 (Maginal rates of substitution)**.** *For $C \in \{A, B\}$ and $(x, y) \in \mathbb{R}^2_+$, we define*

[7] Notice that we extend the domain for u^C from $\mathbb{R}^2_+$ to $\mathbb{R}^2$. This is to avoid some issues with corner solutions in some of the technical parts of the proof of Theorem 1.

[8] Here we define closed relative to the set $\overline{\mathbb{R}}_+$.

the set $\Omega^C(x,y)$ *such that,*[9]

$$\begin{aligned}
&\omega \in \Omega^C(x,y) \cap \mathbb{R}_{++} && \textit{iff } \forall (x',y') \in \mathbb{R}^2 : \omega x' + y' \leq \omega x + y \textit{ implies } u^C(x,y) \geq u^C(x',y'),\\
&+\infty \in \Omega^C(x,y) && \textit{iff } \ \forall (x',y') \in \mathbb{R}^2 : x' \leq x \textit{ implies } u^C(x,y) \geq u^c(x',y'),\\
&0 \in \Omega^C(x,y) && \textit{iff } \ \forall (x',y') \in \mathbb{R}^2 : y' \leq y \textit{ implies } u^C(x,y) \geq u^C(x',y').
\end{aligned}$$

In the special case where u^C is differentiable at (x,y) and $\frac{\partial u^C(x,y)}{\partial y} \neq 0$, the set $\Omega^C(x,y)$ contains the unique marginal rate of substitution $\frac{\partial u^C(x,y)}{\partial x} \Big/ \frac{\partial u^C(x,y)}{\partial y}$. Note however that our definition allows for non-differentiable, vertical or horizontal indifference curves. In particular, when the indifference curves are vertical, preferences are fully determined by the quantity of x, whereas for the case of horizontal indifference curves preferences are determined by the quantity of y.

Let us now define the notion of single crossing indifference curves. Intuitively, if A and B have single crossing indifference curves, it means that A's indifference curves are 'flatter' than B's. If utility functions are differentiable, this can be expressed by the condition that at any point (x,y), B's marginal rate of substitution must be at least as large compared to A's

$$\left(\frac{\partial u^B(x,y)}{\partial x} \Big/ \frac{\partial u^B(x,y)}{\partial y} \right) \geq \left(\frac{\partial u^A(x,y)}{\partial x} \Big/ \frac{\partial u^A(x,y)}{\partial y} \right).$$

Figure 1 illustrates such a case. Given that the the sets $\Omega^A(x,y)$ and $\Omega^B(x,y)$ may contain more than one value, we extend the notion of 'at least as large' to a set-order sense.

Definition 3 (Single crossing indifference curves)**.** *Consumers* A *and* B *are said to have single crossing indifference curves if for all* $(x,y) \in \mathbb{R}^2_+$*,*

$$\Omega^B(x,y) = \Omega^A(x,y) \vee \Omega^B(x,y) \tag{3}$$

and,

$$\Omega^A(x,y) = \Omega^A(x,y) \wedge \Omega^B(x,y). \tag{4}$$

The definition implies that A and B have single crossing indifference curves if for all $(x,y) \in \mathbb{R}^2_+$ and all $\omega^A \in \Omega^A(x,y)$ there is a value $\omega^B \in \Omega^B(x,y)$ such that $\omega^B \geq \omega^A$ and for every $\omega^B \in \Omega^B(x,y)$ there is a value $\omega^A \in \Omega^A(x,y)$ such that $\omega^A \leq \omega^B$.[10]

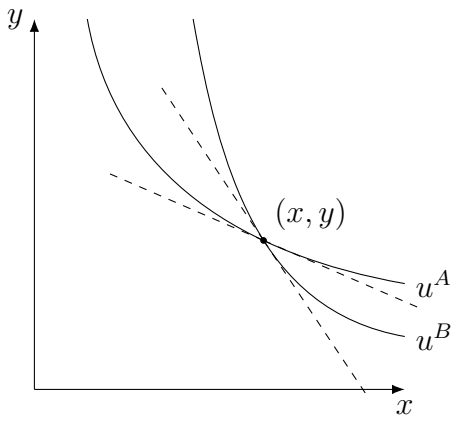
[9] Note that, given monotonicity of utilities, this also gives the strict counterparts:

$$\begin{aligned}
&\omega x' + y' < \omega x + y \text{ implies } u^C(x,y) > u^C(x',y') && \text{for } \omega \in \Omega^C(x,y) \cap \mathbb{R}_{++},\\
&x' < x \text{ implies } u^C(x,y) > u^c(x',y') && \text{for } +\infty \in \Omega^C(x,y),\\
&y' < y \text{ implies } u^C(x,y) > u^C(x',y') && \text{for } 0 \in \Omega^C(x,y).
\end{aligned}$$

[10] We use the convention that $+\infty \geq \omega$ for all $\omega \in \overline{\mathbb{R}}_+$.

Given that the sets $MRS^A(x,y)$ and $MRS^B(x,y)$ conditions (3) and (4) are actually equivalent.

Figure 1: Single crossing indifference curves



Now assume that A and B have single crossing indifference curves in the sense of Definition 3 and they both choose an optimal bundle from the same budget. Intuitively, from Figure 1, it appears that in this case, consumer B will have an optimal choice that contains at least as much of good x compared to consumer A, giving $B \succeq_x A$. As it turns out, the reverse also holds. This is formalized in the following theorem.

Theorem 1. *Assume that utility functions are monotone, quasi-concave and continuous, then consumers A and B have single crossing indifference curves if and only if $B \succeq_x A$.*

3 Revealed preference conditions

We now turn to the testable implications of choice dominance. Assume that we have access to two finite datasets $\mathcal{O}^A = (\omega_t^A, z_t^A, x_t^A, y_t^A)_{t\in T^A}$ and $\mathcal{O}^B = (\omega_v^B, z_v^B, x_v^B, y_v^B)_{v\in T^B}$ containing a finite number of observations T^A and T^B on budgets (ω_t^A, z_t^A) and (ω_t^B, z_t^B) and corresponding choices (x_t^A, y_t^A) and (x_t^B, y_t^B) for consumers A and B. These choices satisfy the budget constraints $z_t^A = \omega_t^A x_t^A + y_t^A$ and $z_t^B = \omega_t^B x_t^B + y_t^B$ for all $t \in T^A$ and $t \in T^B$. Notice that we do not require that T^A and T^B are of equal size nor that the budgets (ω_t^A, z_t^A) and (ω_t^B, z_t^B) are the same.

Our goal is to find conditions on $\mathcal{O}^A$ and $\mathcal{O}^B$ such that there exists utility functions u^B and u^A that can explain the observed choices and such that $B \succeq_x A$. This motivates the following definition.

Definition 4 (Choice dominance rationalizability)**.** *We say that the datasets $\mathcal{O}^A = (\omega_t^A, x_t^A, y_t^A, z_t^A)_{t\in T^A}$ and $\mathcal{O}^B = (\omega_t^B, x_t^B, y_t^B, z_t^B)_{t\in T^B}$ are choice-dominance rationalizable if there exist monotone*

utility functions u^A and u^B, with corresponding (non-empty) demand correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$, such that:

- *$B \succeq_x A$, i.e., for all budgets $(\omega, z) \in \mathbb{R}^2_{++}$:*

$$\mathcal{D}^B(\omega, z) = \mathcal{D}^B(\omega, z) \curlyvee \mathcal{D}^A(\omega, z) \qquad \text{and} \qquad \mathcal{D}^A(\omega, z) = \mathcal{D}^A(\omega, z) \curlywedge \mathcal{D}^B(\omega, z).$$

- *For all observations $t \in T^A$ and $v \in T^B$,*

$$(x_t^A, y_t^A) \in \mathcal{D}^A(\omega_t^A, z_t^A) \qquad \text{and} \qquad (x_v^B, y_v^B) \in \mathcal{D}^B(\omega_v^B, z_v^B).$$

To characterize the datasets that are choice dominance rationalizable we define some revealed preference concepts. For individuals $C, D \in \{A, B\}$ and observations $t \in T^C$ and $v \in T^D$, we say that the bundle (x_t^C, y_t^C) is revealed preferred to the bundle (x_v^D, y_v^D) if

$$z_t^C \geq \omega_t^C x_v^D + y_v^D,$$

In words, (x_t^C, y_t^C) is revealed preferred to (x_v^D, y_v^D) if the expenditure z_t^C at observation $t \in T^C$, which was used to buy (x_t^C, y_t^C), was large enough to also buy the bundle (x_v^D, y_v^D). We denote this by $(x_t^C, y_t^C)\, R\, (x_v^D, y_v^D)$. If observed choices are utility maximizing this implies that $u^C(x_t^C, y_t^C) \geq u^C(x_v^D, y_v^D)$.

We say that (x_t^C, y_t^C) is strictly revealed preferred to (x_v^D, y_v^D) if

$$z_t^C > \omega_t^C x_v^D + y_v^D.$$

We also denote this by $(x_t^C, y_t^C)\, P\, (x_v^D, y_v^D)$. If choices are utility maximizing and if the utility function is monotone this implies $u^C(x_t^C, y_t^C) > u^C(x_v^D, y_v^D)$. Note that if the datasets are choice dominance rationalizable, then we must have that for all $t, v \in T^C$ it is impossible that $(x_t^C, y_t^C)\, R(x_v^C, y_v^C)$ and $(x_v^C, y_v^C)\, P(x_t^C, y_t^C)$, as this would give the preference reversal $u^C(x_t^C, y_t^C) \geq u^C(x_v^C, y_v^C)$ and $u^C(x_v^C, y_v^C) > u^C(x_t^C, y_t^C)$. This gives the well-known Generalized Axiom of Revealed Preference (GARP) in our setting.

Definition 5 (GARP (Afriat, 1967) and (Varian, 1982))**.** *For an individual $C \in \{A, B\}$, we say that the dataset $\mathcal{O}^C = (\omega_t^C, x_t^C, y_t^C, z_t^C)_{t \in T^C}$ satisfies the Generalized Axiom of Revealed Preference (GARP) if for all observations $t, v \in T^C$,*

$$(x_t^C, y_t^C)\, R(x_v^C, y_v^C) \text{ implies that not } (x_v^C, y_v^C)\, P(x_t^C, y_t^C)$$

Usually, GARP uses the transitive closure of R instead of the relation R. (the so called indirect revealed preference relation). In the two goods setting, however, this is not necessary, so we obtain the simpler condition as stated in Definition 5.[11]

As shown by Afriat (1967) and Varian (1982) GARP is a necessary and sufficient condition on a dataset to be rationalizable in the sense that there exists a monotone utility function

[11] See Rose (1958), Banerjee and Murphy (2006) and Cherchye, Demuynck and De Rock (2018).

u^C with corresponding demand correspondence $\mathcal{D}^C$ satisfying $(x_t^C, y_t^C) \in \mathcal{D}^C(\omega_t^C, z_t^C)$ for all $t \in T^C$.

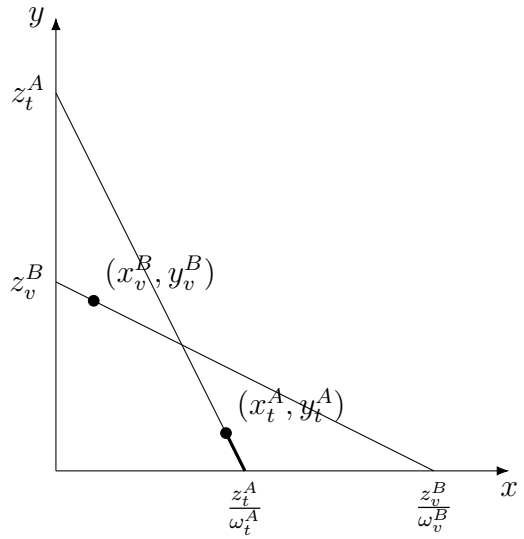
For choice-dominance rationalizability we require also that $B \succeq_x A$. This additional requirement imposes additional restrictions on the two datasets $\mathcal{O}^A$ and $\mathcal{O}^B$. This is illustrated in Figure 2. The figure depicts a budget set (ω_t^A, z_t^A) for individual A with choice (x_t^A, y_t^A) and a budget set (ω_t^B, z_t^B) for individual B with observed choice (x_v^B, y_v^B). Note that there is a GARP-type violation in the sense that:

$$(x_t^A, y_t^A)\, R(x_v^B, y_v^B) \text{ and } (x_v^B, y_v^B)\, P(x_t^A, y_t^A).$$

Note, however, that this doesn't give a GARP violation, in the sense of Definition 5, as the two choices are from different individuals.

In addition to the GARP-type violation, the two budgets in the figure are such that the relative price of x for individual B is less than the relative price of x for individual A: $\omega_v^B < \omega_t^A$. One easily sees that this data is not consistent with $B \succeq_x A$. Towards a contradiction assume that $B \succeq_x A$. Then from the budget (ω_t^A, z_t^A) individual B has an optimal choice $(\hat{x}_t^B, \hat{y}_t^B)$ with $\hat{x}_t^B \geq x_t^A$. Any such bundle lies on the thicker part of the budget line, i.e., the segment to the right of (x_t^A, y_t^A). But any such choice leads to a violation of GARP for B with the observed choice (x_v^B, y_v^B) from budget (ω_v^B, z_v^B). In particular we have that $(\hat{x}_t^B, \hat{y}_t^B)\, P(x_v^B, y_v^B)$ and $(x_v^B, y_v^B)\, P(\hat{x}_t^B, \hat{y}_t^B)$ which produces the preference reversal: $u^B(\hat{x}_t^B, \hat{y}_t^B) > u^B(x_v^B, y_v^B)$ and $u^B(x_v^B, y_v^B) > u^B(\hat{x}_t^B, \hat{y}_t^B)$. A similar reasoning holds for consumer A.

Figure 2: Revealed preference restriction with single crossing indifference curves



Motivated by this example, we introduce the following revealed preference condition, called X-GARP.

Definition 6 (X-GARP)**.** *The datasets* $\mathcal{O}^A = (\omega_t^A, x_t^A, y_t^A, z_t^A)_{t\in T^A}$ *and* $\mathcal{O}^B = (\omega_t^B, x_t^B, y_t^B, z_t^B)_{t\in T^B}$ *are said to satisfy* X*-GARP if, for all observations* $t \in T^A$ *and* $v \in T^B$ *with* $\omega_v^B \leq \omega_t^A$*:*

$$\text{if } (x_t^A, y_t^A)\, R\, (x_v^B, y_v^B) \text{ then not } (x_v^B, y_v^B)\, P\, (x_t^A, y_t^A).$$

and

$$\text{if } (x_v^B, y_v^B)\, R\, (x_t^A, y_t^A) \text{ then not } (x_t^A, y_t^A)\, P\, (x_v^B, y_v^B).$$

The following main result provides the characterisation for two datasets $\mathcal{O}^A$ and $\mathcal{O}^B$ to be consistent with choice-dominance rationalizability.

Theorem 2. *Consider two data sets* $\mathcal{O}^A = (\omega_t^A, x_t^A, y_t^A, z_t^A)_{t\in T^A}$ *and* $\mathcal{O}^B = (\omega_t^B, x_t^B, y_t^B, z_t^B)_{t\in T^B}$*. Then the following conditions are equivalent.*

i. There are monotone utility functions u^A *and* u^B *with non-empty demand correspondences* $\mathcal{D}^A$ *and* $\mathcal{D}^B$ *that choice dominance rationalize the data sets* $\mathcal{O}^A$ *and* $\mathcal{O}^B$*.*

ii. The data sets $\mathcal{O}^A$ *and* $\mathcal{O}^B$ *separately satisfy GARP and jointly satisfy* X*-GARP*

iii. There are monotone and quasi-concave utility functions u^A *and* u^B *with non-empty demand correspondences* $\mathcal{D}^A$ *and* $\mathcal{D}^B$ *that choice dominance rationalize the datasets* $\mathcal{O}^A$ *and* $\mathcal{O}^B$*.*

Theorem 2 shows that the joint condition of GARP and X-GARP exhaust all restrictions on a collection of observed choices from a pair of consumers to be choice dominance rationalizable. The equivalence between *(i)* and *(iii)* highlights the fact that quasi-concavity itself cannot be tested in this setting. That *(iii)* implies *(i)* is straightforward. The intuition that *(i)* implies *(ii)* was given in Figure 2. The most difficult (and longest) step is proving that (*ii*) implies (*iii*). For this proof, we abstain from explicitly constructing the underlying utility functions u^A and u^B that satisfy Definition 1. Instead we focus on constructing two demand correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$ such that:

A) Definition 1 (preference dominance) is satisfied,

B) for all for all $t \in T^A$ and $v \in T^B$,

$$(x_t^A, y_t^A) \in \mathcal{D}^A(\omega_t^A, z_t^A),$$
$$(x_v^B, y_v^B) \in \mathcal{D}^B(\omega_v^B, z_v^B).$$

C) they coincide with the demand correspondences of some quasi-concave and monotone utility functions u^A and u^B.

The proof of Theorem 2 first establishes an intermediate result to tackle C). Then it provides a constructive proof for parts A) and B).

4 Illustrative example

To illustrate the practical relevance of our X-GARP condition, we use the experimental data of Porter and Adams (2015). In this experiment 190 subjects played two rounds of modified dictator games. One round was with a parent and one was with a stranger as recipients. For each dictator game the total budget to allocate and the price of giving varied. Each participant played a total of 11 games in each round and in each game the subject was was asked to divide tokens by filling in a form comprised of questions of the type "*Divide 50 tokens: Hold ____ tokens at 10 pence each, and Pass ____ tokens at 20 pence each.*"[12]

We match our theoretical framework to this experimental setting in the following way. We consider two types of individuals: a subject is of type S(tranger) when allocating tokens between herself and a stranger and of type P(arent) when she is playing with a parent. In each round she spends her budget over two goods: 'keeping' (x) and 'giving' (y). For each participant we have a pair of datasets $\mathcal{O}^S$ and $\mathcal{O}^P$ corresponding to the budgets and chosen allocations in the 11 games that she played with a stranger and with a parent. The revealed preference restrictions in Theorem 3 allow us to determine whether individuals' choices in each round satisfy GARP and additionally if subjects tend to keep more money when the recipient is a stranger than when she is a parent. Our hypothesis is that the S-type $\succeq_x$ dominates the P-type.

$$H_0 : S \succeq_x P$$

As alternative hypothesis we can take $H_1 : P \succeq_x S$. Note that we can check the revealed preference condition separately for each subject in the experiment so there is no need to pool data across different subjects. Table 1 gives the pass rates of the various tests. Column 3 gives the fraction of subjects for which both datasets $\mathcal{O}^S$ and $\mathcal{O}^P$ pass GARP. Column 4 gives the pass rate for the X-GARP test. The last column imposes both GARP and X-GARP simultaneously. The rows correspond to the two hypothesis.

We see that more than 80 percent of subjects behave in a way consistent with utility maximization both when interacting with a stranger and with a parent. The third column shows that also more than 80 percent of the subjects satisfy the X-GARP test for $S \succeq_x P$ while only 38% satisfies the test for $P \succeq_x S$.[13] The final column shows that for about 75% of the subjects we can not reject the hypothesis that both types are rational and that the S-type x-dominates the P-type. On the other hand for only 35 percent of the subjects we can not reject the hypothesis that both types are rational and that the P type x-dominates the S type. The difference in mean pass rates between the two settings is significant with ($p < 0.001$) according to a McNemar's test. We therefore strongly support

[12] In the original study Porter and Adams (2015) allocated subjects to three treatment groups according to the level of information parents had about the game played in the experiment. The results of the analysis carried out separately for each treatment group are available in Appendix B.2. The pass rates do not differ significantly across treatment groups according to a Pearson's Chi-square test.

[13] In appendix B.1 we also report on the share of participants who statisfy X-GARP conditional on passing GARP.

Table 1: Pass rates and power for Theorem 2

Setting	Variable	GARP	X-GARP	GARP & X-GARP
$S \succeq_x P$	Pass rate	0.832	0.858	0.742
$P \succeq_x S$	Pass rate	0.832	0.384	0.353
	Power	0.996	0.912	0.998

our null-hypothesis.

For completeness we also assess the stringency of the tests by means of a Bronars power index (Bronars, 1987). The Bronars index quantifies how demanding the revealed preference conditions are by computing the pass rates for subjects behaving irrationally. Irrational behaviour is here simulated by generating random choice bundles within a given budget for a certain price regime. To be precise, we generated 1,000 random subjects by drawing for each of them 2 random allocations of 'keeping' and 'giving' at each of the 11 budget-price combinations. We then compute the various pass rates for these randomly generated subjects. Reassuringly, we find that more than 90% of the simulated subjects violate the tests implying that our test has an important discriminatory power.

5 Choice dominance in collective households

A large literature in economics concerns the evaluation of large-scale (conditional) cash transfer programmes. These programmes provide money to impoverished families based on certain conditions such as sending their children to school. They have found to be effective in reducing poverty, supporting consumption, improving preventive healthcare, and enhancing human capital. A significant question related to these transfers is whether they should be granted to the adult females in the household. The underlying rationale is that if the mother receives the transfer, then the outcomes, especially for children, are expected to be more positive compared to if the father receives it. The reasoning relies on two assumptions. First, by shifting the income distribution within the household towards women this increases their empowerment and bargaining power. Second, women have a stronger preference to allocate resources towards goods that benefit their children compared to their husbands.[14]

In this section we provide a theoretical basis for this argument by integrating the concept of choice dominance into a simple collective household model. We then derive a comparative statics results arising from an increase in the bargaining power for one of the household

[14]The empirical evidence supporting the overall effect varies. While some find supportive evidence (for example, Armand, Attanasio, Carneiro and Lechene (2020)) others obtain more mixed findings (for example, IEG (2014), Benhassine, Devoto, Duflo, Dupas and Pouliquen (2015), Bastagli, Hagen-Zanker, Harman, Barca, Sturge, Schmidth and Pellerano (2016), (Haushofer and Shapiro, 2016); Ringdal and Home Sjursen (2021), Cherchye, Chiappori, De Rock, Ringdal and Vermeulen (2021)).

members. We use this to derive an X-GARP type restriction on the observable household consumption after a shift in the bargaining power of one household member.

Consider a household with two members A (the father) and B (the mother) that consumes a bundle $(x, y) \in \mathbb{R}^2_+$. We follow the collective household literature and assume that household choices are Pareto optimal in the sense that there is some utility level $\bar{u}$ such that the household demand correspondence $\mathcal{D}$ is determined by the solutions to the following optimization problem.

$$\mathcal{D}(\omega, z, \bar{u}) = \arg\max_{x,y\geq 0} u^A(x, y) \text{ s.t. } \begin{cases} \omega x + y \leq z, \\ u^B(x, y) \geq \bar{u}. \end{cases}$$

Here ω represents the (relative) price of x and z is the normalized household income. The value $\bar{u}$ is the reservation utility for member B. Its value encompasses the outside option for B. For example, $\bar{u}$ can be determined by the utility achieved in a non-cooperative outcome between A and B or it can represent the utility B can obtain upon divorce. Usually it is assumed that this reservation utility is influenced by variables that are often referred to as *distribution factors* in the literature (Chiappori and Mazzocco, 2017).[15] In short $\bar{u}$ could be interpreted as the (relative) bargaining power of individual B in the sense that an increase in $\bar{u}$ generally implies an increase in the final utility received by this individual.

Note that the bundle (x, y) enters both the utility function of A and B. This assumption is satisfied if x and y are both household-level public goods or if they are private but there are positive externalities associated with their consumption. A second case where this assumption is satisfied is when x and y are private but there is some fixed (predetermined) division rule between the household members: there are some fixed numbers $\alpha, \beta \in (0, 1)$ such that A consumes a fraction $(\alpha x, \beta y)$ and B consumes the residual fraction $((1 - \alpha) x, (1 - \beta) y)$ of the bundle (x, y).

To link this setting with our idea of choice dominance, we assume that B choice dominates A in terms of good x: $B \succeq_x A$. Now, consider two choice situations with identical budgets (ω, z), but where the second has a higher value of $\bar{u}$ compared to the first one. Intuitively this corresponds to a case where the bargaining power of B increased. If also $B \succeq_x A$, we would expect that, as a result of this increase in bargaining power, the final household consumption bundle should now contain more of good x.

The following theorem shows that this intuition is correct.

Theorem 3. *Assume that u^A and u^B are continuous, monotone and quasi-concave and that $B \succeq_x A$. Let $\bar{u}' > \bar{u}$, and assume that $\mathcal{D}(\omega, z, \bar{u})$ and $\mathcal{D}(\omega, z, \bar{u}')$ are both non-empty. Then,*

$$\mathcal{D}(\omega, z, \bar{u}') = \mathcal{D}(\omega, z, \bar{u}) \curlyvee \mathcal{D}(\omega, z, \bar{u}').$$

or equivalently,

$$\mathcal{D}(\omega, z, \bar{u}) = \mathcal{D}(\omega, z, \bar{u}) \curlywedge \mathcal{D}(\omega, z, \bar{u}').$$

[15] Typical examples are relative potential earnings, divorce legislation or the sex ratio in (local) marriage markets etc. See Chiappori and Mazzocco (2017) for a further discussion of their use in the literature.

In words Theorem 3 confirms the intuition that if the bargaining power of B increases, then for a given budget the household's demand correspondence will always contain consumption bundles that contain at least as much of good x.

The next theorem shows that Theorem 3 has empirical bite: if we observe two household choices with budgets (ω, z) and (ω', z') and threshold utility levels $\bar{u}$ and $\bar{u}'$ where $\bar{u}' > \bar{u}$, then these observations must satisfy our X-GARP condition.

Theorem 4. *Assume that the assumptions of Theorem 3 are satisfied. Let* $(x, y) \in \mathcal{D}(\omega, z, \bar{u})$ *and* $(x', y') \in \mathcal{D}(\omega', z', \bar{u}')$ *with* $\bar{u}' > \bar{u}$. *Then if* $\omega' \leq \omega$ *it must be that,*

$$(x, y)\, R(x', y') \text{ implies that not } (x', y')\, P(x, y),$$

and,

$$(x, y)\, P(x', y') \text{ implies that not } (x', y')\, R(x, y).$$

In order test the X-GARP condition in Theorem 4 we need at least two observations (x, y) and (x', y') from two household budgets (ω, z) and (ω', z') and associated threshold utility levels $\bar{u}$ and $\bar{u}'$ such that $\omega' \leq \omega$ and $\bar{u} < \bar{u}'$.

In reality the values of $\bar{u}$ and $\bar{u}'$ are not directly observable. In practice one might therefore look at a setting where one has a clear indication that the relative bargaining power for one of the spouses has unambiguously increased. One such example could be an (exogenous) change in the divorce law that changes the way total household resources are split at divorce.[16] If such law change is more favourable to one of the spouses this may induce a large enough increase in $\overline{u}$. Another example could be a conditional cash transfer given to the female household member instead of the male household member. The majority of cash transfer programs indeed designate the mother in the household as the recipient. This is the setting we turn to next.

6 Application

We use the PROGRESA data to investigate whether households shift their dietary choices towards more nutritious options in response to monetary transfers to the mother. PROGRESA is a governmental social assistance program which was implemented in rural Mexico in the late 1990s. It provided families with cash payments conditional on their children attending school.

Our analysis is motivated by some previous findings that PROGRESA enabled households to gradually increase their caloric intake with the program's most significant impact observed in the consumption of highly nutritious foods, such as fruits, vegetables, and animal products (Hoddinott, Skoufias and Washburn, 2000; Hoddinott and Skoufias, 2004; Angelucci and Garlick, 2015).

[16] An example is the White v. White case in England (De Rock, Kovaleva and Potoms, 2023).

We rely on two post-intervention surveys of the PROGRESA program conducted respectively six and twelve months after households in treated villages started receiving cash transfers in October 1998 and June 1999. Households in control villages did not start receiving transfers until after these two surveys. Previous research indicates that the program's effects on dietary choices emerged relatively late typically around 12 months after transfers began (Hoddinott et al., 2000; Skoufias, 2005). The data collected contains information on household composition, members' characteristics, and consumption, with detailed records of expenditure on a variety of food items. We focus on households with one to six children where both parents are present. We exclude households in which some members either do not live in the household but generally eat there or live in the household but generally do not eat there.

Since our data does not provide a unique household identifier across survey waves we matched households based on the following characteristics: village, gender of the household head, whether the head speaks an indigenous language, education levels of the head and spouse, number of children, and the household head's age (in five-year intervals). When multiple households within a wave had identical values for all these characteristics we randomly selected one of them. To prevent any sensitivity in the results that might arise from this random selection we generated 100 random datasets using this matching procedure.

Our data includes prices and monthly expenditure on five categories of food commodities: (1) fruits and vegetables, (2) meat, fish and dairy, (3) starches, (4) pulses, and (5) other foods. We aggregate these into two broader categories: highly nutritious (h) food ((1) and (2)), and less nutritious (ℓ) food ((3), (4), and (5)). Aggregated prices are calculated as geometric averages of commodity prices with the weight of each commodity determined by its share of expenditure in the respective food group. Quantities are computed by dividing total household expenditure on these aggregates by the constructed prices.

Table 4 in Appendix C presents some summary statistics for households in the treatment and control groups and shows that they are balanced across a range of relevant characteristics in the first wave of the survey. Food expenditures do differ between the groups: households in treated villages spend more overall on both highly nutritious and less nutritious food. Importantly for our analysis, the ratio of expenditure on highly nutritious to less nutritious food remains similar between the groups. This difference in expenditure is not surprising as the first survey wave was collected six months after the transfers began allowing time for treated households to gradually increase their food consumption.

Results and discussion Our analysis focuses on whether PROGRESA increased households' consumption for highly nutritious (h) food relative to less nutritious (ℓ) food. Each household chooses the quantities of h (x) and ℓ food (y) that maximize their utility. We denote the relative prices in the first wave (October 1998) and second wave (June 1999) of the post-treatment survey by $\omega^{98} = \frac{p_h^{98}}{p_\ell^{98}}$ and $\omega^{99} = \frac{p_h^{99}}{p_\ell^{99}}$, respectively, and the corresponding household choices of h and ℓ by $q^{98} = (h^{98}, \ell^{98})$ and $q^{99} = (h^{99}, \ell^{99})$.

It is reasonable to assume that PROGRESA improved women's bargaining power within their homes by increasing the share of household resources they control (De Rock, Potoms and Tommasi, 2022). In that case, Theorem 3 tells us that if a woman's bargaining power in treated households increased between the first and second wave and if (within a household) the woman choice dominate the man for h-food, then the household should consume relatively more h food in the second wave of the survey, conditional on the budget. Theorem 4 then tells us that the joint data $(\omega^{98}, q^{98}, \omega^{99}, q^{99})$ in the treatment group should satisfy X-GARP. Given that for the control households we do not expect an unambiguous increase in the bargaining power of the mother or father between the two waves, we expect the pass rate of the X-GARP condition to be lower in the control group.

Note that our test is fairly weak as we have per household only two observations to test for X-GARP.[17] Also note that for the X-GARP condition to be falsifiable it is necessary that the budget lines from the two periods intersect and that prices vary in the right direction: $\omega^{99} \leq \omega^{98}$. If one of these conditions is not met, X-GARP will automatically be satisfied. For this reason we only compute the pass rates as a fraction of all observations meeting the corresponding price ratio condition and for which the budget lines from the two periods intersect. This gives a total of 200 households in the treatment sample and 220 in the control group.

For the treatment group we find that on average 94.4 percent of these households satisfy the X-GARP condition.[18] In the control group, the average drops to 89.8 percent. This means that the pass rate for the X-GARP condition is on average 4.6 percentage points higher in treated compared to control households. This difference is significant at the 10 per cent level as measured by a two-proportion z-test ($p = 0.069$). This finding is consistent with existing literature suggesting that PROGRESA led treated households to shift their consumption towards a more nutritious diet.

7 Conclusion

We introduced and studied a behavioural condition called choice dominance which allows us to compare individuals based on who consumes more of a particular good. We relate this condition to a Spence-Mirrlees single-crossing restriction on the indifference curves of the two individuals.

The main contribution of our paper provides a non-parametric revealed preference characterisation of choice dominance. We show that beyond the classical Generalized Axiom of Revealed Preference (GARP) condition on the choices of both individuals separately, we need a condition referred to as X-GARP. This condition imposes the absence of a GARP violation in the pooled consumption choices for both individuals when B is confronted with

[17] The test would become much more powerful if we would have multiple observations before and after the intervention.

[18] Averaged over the 100 'random' datasets generated as described earlier

a lower relative price for good x compared to A.

We then apply the notion of choice dominance to a simple household model. We show that if a household member B choice dominates member A for good x and when the bargaining power of B increases then the household's demand correspondence will always contain consumption bundles that contain as least as much of good x (for a given budget). Theorem 4 provides some testable implications: if we observe two household choices with budgets (ω, z) and (ω', z') and threshold utility levels $\bar{u}$ and $\bar{u}'$ respectively, then if $\bar{u}' > \bar{u}$, these observations should satisfy our X-GARP condition.

Two empirical applications demonstrate the usefulness of our results.

An important follow-up question for future work would be to see how our framework could be generalized to settings with more than two goods. Tackling a full characterisation for such situations will most likely require a new approach as the proof of the main result in this paper crucially relies on the construction of a rational income-expansion paths, which are, in the two goods case more easily manageable (i.e., one dimensional). An additional difficulty is that in the more-than-two-goods case the usual GARP conditions need to take into account the indirect revealed preference relation (i.e., transitivities).

Other extensions to the current work could focus on other types of preference comparisons. One comparison that comes to mind is the notion of risk aversion. The revealed preference literature has several models that look at choices under risk or uncertainty (Green and Osband, 1991; Echenique and Saito, 2015; Polisson, Quah and Renou, 2020; Demuynck and Staner, 2024). It would be interesting to see how these models could be used or extended to test whether an individual is more risk averse compared to another one.

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A Proofs

A.1 Proof of theorem 1

Proof. ($\rightarrow$) Assume that the single crossing condition holds. As indifference curves are quasi-concave the correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$ are convex valued. This means that we only need to demonstrate either (1) or (2) as both are equivalent. We focus on demonstrating (1).

Let us first show that $\mathcal{D}^A(\omega, z) \curlyvee \mathcal{D}^B(\omega, z) \subseteq \mathcal{D}^B(\omega, z)$. Consider a budget (ω, z) and let $(x^A, y^A) \in \mathcal{D}^A(\omega, z)$ and $(x^B, y^B) \in \mathcal{D}^B(\omega, z)$.

If $x^B \geq x^A$ then $(x^A, y^A) \curlyvee (x^B, y^B) = (x^B, y^B) \in \mathcal{D}^B(\omega, z)$ which we needed to demonstrate.

Now consider the case where $x^B < x^A$ (and $y^B > y^A$). Then $(x^A, y^A) = (x^A, y^A) \curlyvee (x^B, y^B)$, so we need to show that $(x^A, y^A) \in \mathcal{D}^B(\omega, z)$. Towards a contradiction assume that $(x^A, y^A) \notin \mathcal{D}^B(\omega, z)$.

From optimality of (x^A, y^A) for A under budget (ω, z) we have that $\omega \in \Omega^A(x^A, y^A)$. Let $\omega' \in \Omega^B(x^A, y^A)$ with $\omega' \geq \omega$ which exists from the single crossing condition. If $\omega' = +\infty$, we have that $x^A > x^B$ implies $u^B(x^A, y^A) > u^B(x^B, y^B)$. This contradicts the assumption that $(x^B, y^B) \in \mathcal{D}^B(\omega, z)$ and $(x^A, y^A) \notin \mathcal{D}^B(\omega, z)$ so (x^B, y^B) could not be utility maximising for B at budget (ω, z). If $\omega' = \omega$ then $u^B(x^A, y^A) \geq u^B(x, y)$ for all (x, y) that satisfy $\omega x + y \leq \omega x^A + y^A = z$. Then by definition $(x^A, y^A) \in \mathcal{D}^B(\omega, z)$ which is a contradiction.

We conclude that $\omega' \in \mathbb{R}_{++}$ and $\omega' > \omega$. As $u^B(x^B, y^B) > u^B(x^A, y^A)$ (as $(x^A, y^A) \notin \mathcal{D}^B(\omega, z)$), it follows that by contrapositive of Definition 2:

$$\omega' x^A + y^A < \omega' x^B + y^B.$$

We also have that $\omega x^B + y^B = \omega x^A + y^A = z$. Subtracting this from the previous inequality gives,

$$(\omega' - \omega)(x^A - x^B) < 0.$$

The first term is strictly positive so it must be that $x^A < x^B$ which gives the contradiction.

To see that $\mathcal{D}^B(\omega, z) \subseteq \mathcal{D}^A(\omega, z) \curlyvee \mathcal{D}^B(\omega, z)$ let $(x^B, y^B) \in \mathcal{D}^B(\omega, z)$ and assume towards a contradiction that $(x^B, y^B) \notin \mathcal{D}^A(\omega, z) \curlyvee \mathcal{D}^B(\omega, z)$. This means that for all $(x^A, y^A) \in \mathcal{D}^A(\omega, z)$, $x^A > x^B$. Note also that $\omega \in \Omega^B(x^B, y^B)$. From the single crossing condition there is an $\omega^A \in \Omega^A(x^B, y^B)$ such that $\omega^A \leq \omega$. Then:

$$\omega^A x^B + y^B = \omega^A x^B + z - \omega x^B = \underbrace{(\omega^A - \omega)}_{\leq 0} \underbrace{x^B}_{< x_A} + z \geq (\omega^A - \omega)x^A + z = \omega^A x^A + y^A$$

As $\omega^A \in \Omega^A(x^B, y^B)$ it follows that $u^A(x^B, y^B) \geq u^A(x^A, y^A)$. From $(x^A, y^A) \in \mathcal{D}^A(\omega, z)$ it follows that $(x^B, y^B) \in \mathcal{D}^A(\omega, z)$ which gives the contradiction.

($\leftarrow$) Assume that $B \succeq_x A$. We want to show that the single crossing condition holds: $\Omega^B(x, y) = \Omega^A(x, y) \vee \Omega^B(x, y)$.

To see that $\Omega^A(x, y) \vee \Omega^B(x, y) \subseteq \Omega^B(x, y)$ consider $\omega^A \in \Omega^A(x, y)$ and $\omega^B \in \Omega^B(x, y)$. If $\omega^B \geq \omega^A$ then $\omega^B \vee \omega^A = \omega^B \in \Omega^B(x, y)$ so we are done. As such assume that $\omega^B < \omega^A$.

Let us show that $\omega^A \in \Omega^B(x, y)$. If $\omega^A < +\infty$, then from a simple revealed preference argument, we have that $(x, y) \in \mathcal{D}^A(\omega^A, z)$ where $z = \omega^A x + y$. Let $(x^B, y^B) \in \mathcal{D}^B(\omega^A, z)$ where $x^B \geq x$, which exists as $B \succeq_x A$.

Notice that $u^B(x^B, y^B) \geq u^B(x, y)$. So

$$\omega^B x + y \leq \omega^B x^B + y^B,$$

Also by definition $\omega^A x + y = \omega^A x^B + y^B$. Subtracting this equation from the previous inequality gives:

$$(\omega^B - \omega^A)(x - x^B) \leq 0.$$

The first term is strictly negative, which means that $x \geq x^B$. But this means that $x = x^B$ (and $y = y^B$). Conclude that $(x, y) \in \mathcal{D}^B(\omega^A, z)$ which implies that $\omega^A \in \Omega^B(x, y)$.

Now consider the case where $\omega^A = +\infty$. We want to show that $+\infty \in \Omega^B(x, y)$. In particular that $u^B(x, y) \geq u^B(x', y')$ for all $x' \leq x$. Let (x', y') satisfy $x' \leq x$. If $y' \leq y$ then immediately from monotonicity of u^B, $u^B(x, y) \geq u^B(x', y')$. As such consider the case where $y' > y$.

Let $((x_t, y_t))_{t\in\mathbb{N}}$ be a bounded sequence of bundles in $\mathbb{R}^2$ such that for all t: $x_t < x$, $y_t > y$ and (x_t, y_t) converges to (x', y'). Let $(|x_t|, |y_t|) \leq (M, M)$ with $M \in \mathbb{R}_+$.

Let $((\omega_t, z_t))_{t\in\mathbb{N}}$ be a sequence of budgets such that ω_t converges to $+\infty$ and $\omega_t x_t + y_t < z_t = \omega_t x + y$. For this it suffices to take $\omega_t > \frac{y_t - y}{x - x_t} > 0$. Without loss of generality, assume that ω_t is non-decreasing and $\omega_t \geq 1$.

For all t we can consider bundles $(x_t^A, y_t^A) \in \mathcal{D}^A(\omega_t, z_t)$ and $(x_t^B, y_t^B) \in \mathcal{D}^B(\omega_t, z_t)$. The budgets (ω_t, z_t) contain the bundle (x, y), so it intersects with $\mathbb{R}^2_+$. As $\Omega^A(x, y) = +\infty$, we can pick bundles (x_t^A, y_t^A) such that $x_t^A \geq x$ (if not, then $u^A(x, y) \geq u^A(x_t^A, y_t^A)$ so $(x, y) \in \mathcal{D}^A(\omega_t, z_t)$ and we can set $(x_t^A, y_t^A) = (x, y)$). As $B \succeq_x A$ we can also pick $x_t^B \geq x_t^A \geq x$. Notice that for all t, $x_t^B \in [x, z_t/\omega_t] \subset [x, 2B]$ and $y_t^B \in [0, y]$. As such both sequences are bounded and therefore have a convergent subsequence converging to a limit which we denote by (x^B, y^B). By a simple revealed preference argument, we have that for all t: $u^B(x_t^B, y_t^B) > u^B(x_t, y_t)$. By continuity of the utility function $u^B(x^B, y^B) \geq u^B(x', y')$. The value x_t^B converges to x (as z_t/ω_t converges to x) so $x^B = x$ and $y^B \in [0, y]$. This means that $(x^B, y^B) \leq (x, y)$ so $u^B(x, y) \geq u^B(x^B, y^B)$. By transitivity, $u^B(x, y) \geq u^B(x^B, y^B)$ as we wanted to show.

Now let us show that $\Omega^B(x, y) \subseteq \Omega^A(x, y) \vee \Omega^B(x, y)$. Let $\omega^B \in \Omega^B(x, y)$. We need to show that $\omega^B \in \Omega^A(x, y) \vee \Omega^B(x, y)$. If $\omega^B = +\infty$, there is nothing to be proven. As such, assume $\omega^B \in \mathbb{R}_+$. Towards a contradiction assume that for all $\omega^A \in \Omega^A(x, y)$, $\omega^A > \omega^B$.

Then $(x, y) \in \mathcal{D}^A(\omega^A, z)$ where $z = \omega^A x + y$. From the choice dominance condition there is an $(x^B, y^B) \in \mathcal{D}^B(\omega^A, z)$ such that $x^B \geq x$. Then:

$$\omega^B x + y = \omega^B x + (z - \omega^A x) = \underbrace{(\omega^B - \omega^A)}_{<0} \underbrace{x}_{\leq x^B} + z \geq (\omega^B - \omega^A)x^B + z = \omega^B x^B + y^B$$

As $\omega^B \in \Omega^B(x, y)$ it follows that $u^B(x, y) \geq u^B(x^B, y^B)$. As $(x^B, y^B) \in \mathcal{D}^B(\omega^A, z)$ we have that $(x, y) \in \mathcal{D}^B(\omega^A, z)$. This in turn implies that $\omega^A \in \Omega^B(x, y)$ which gives the contradiction.

□

A.2 Proof of Theorem 2

Proof. That condition (iii) implies condition (i) is obvious. To see that condition (i) implies condition (ii), assume that $B \succeq_x A$. That $\mathcal{O}^A$ and $\mathcal{O}^B$ need to satisfy GARP is a standard result in revealed preference theory (see, for example Varian (1982)). So, let us focus on showing the other condition. Let $\omega_v^B < \omega_t^A$ and assume, towards a contradiction, that $(x_t^A, y_t^A)\, R(x_v^B, y_v^B)$ and $(x_v^B, y_v^B)\, P(x_t^A, y_t^A)$. This is equivalent to the conditions that:

$$z_t^A \geq \omega_t^A x_v^B + y_v^B \text{ and } z_v^B > \omega_t^B x_t^A + y_t^A.$$

If $x_t^A = x_v^B$ then we have that,

$$y_t^A = z_t^A - \omega_t^A x_t^A = z_t^A - \omega_t^A x_v^B \geq y_v^B$$

and

$$y_v^B = z_v^B - \omega_v^B x_v^B = z_v^B - \omega_v^B x_t^A > y_t^A,$$

which gives a contradiction.

Now, let us consider the case where $x_t^A \neq x_v^B$. Using the budget constraints $y_v^B = z_v^B - \omega_v^B x_v^B$ and $y_t^A = z_t^A - \omega_t^A x_t^A$, we can rewrite the two inequalities that give the violation in the following way:

$$z_t^A \geq z_v^B + (\omega_t^A - \omega_v^B) x_v^B \text{ and } z_v^B > z_t^A + (\omega_v^B - \omega_t^A) x_t^A.$$

Adding these two inequalities together gives:

$$0 > (\omega_t^A - \omega_v^B)(x_v^B - x_t^A).$$

By assumption, the first term on the left hand side is strictly positive, so we have $x_v^B < x_t^A$. Next, let (x^B, y^B) be the optimal choice of individual B from budget (ω_t^A, z_t^A). As $B \succeq_x A$, we may assume that $x^B \geq x_t^A$. Then:

$$z_v^B > z_t^A + \underbrace{(\omega_v^B - \omega_t^A)}_{<0} x_t^A \geq z_t^A + (\omega_v^B - \omega_t^A) x^B$$

As $z_t^A = \omega_t^A x^B + y^B$, this gives:

$$z_v^B > w_v^B x^B + y^B.$$

Also, remember that:

$$z_t^A \geq \omega_t^A x_v^B + y_v^B.$$

The last inequality shows that while (x^B, y^B) was optimal at the budget (ω_t^A, z_t^A), the bundle (x_v^B, y_v^B) was also affordable. By a simple revealed preference argument this means that $u^B(x^B, y^B) \geq u^B(x_v^B, y_v^B)$. The second to last inequality shows that although (x_v^B, y_v^B) was optimal at (ω_v^B, z_v^B) the bundle (x^B, y^B) was strictly cheaper. Again by a simple revealed

preference argument (and monotonicity of u^B), we get that $u^B(x_v^B, y_v^B) > u^B(x^B, y^B)$. We obtain a contradiction.

The proof that condition (ii) implies condition (iii) is somewhat more elaborate. Instead of explicitly constructing the underlying utility functions u^A and u^B and show that the corresponding demand correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$ satisfy definition 1, we will use a instead focus on explicitly constructing the two demand correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$ for A and B, such that:

A) Definition 1 (preference dominance) is satisfied,

B) for all for all $t \in T^A$ and $v \in T^B$,

$$(x_t^A, y_t^A) \in \mathcal{D}^A(\omega_t^A, z_t^A), \qquad \text{and} \qquad (x_v^B, y_v^B) \in \mathcal{D}^B(\omega_v^B, z_v^B).$$

C) they coincide with the demand correspondences of some quasi-concave and monotone utility functions u^A and u^B.

We will start with part C), establishing a collection of sufficient conditions on a demand correspondence $\mathcal{D}^C$ to be the demand correspondence for some quasi-concave and monotone utility function u^C.

Before going there, we generalize the notion of GARP from a property on finite datasets to a property on demand correspondences.

Definition 7 (GARP for correspondences)**.** *We say that a demand correspondence $\mathcal{D}^C$ satisfies GARP if for any two budgets (ω, z) and $(\omega', z') \in \mathbb{R}^2_{++}$ and any two bundles $(x, y) \in \mathcal{D}^C(\omega, z)$ and $(x', y') \in \mathcal{D}^C(\omega', z')$.*

$$(x, y)\, R(x', y') \text{ implies that not } (x', y')\, P(x, y).$$

Here we define $(x, y)\, R(x', y')$ in the usual way by the inequality $z \geq \omega x' + y'$ and we define $(x', y')\, P(x, y)$ by the inequality $z' > \omega' x + y$.

We now state our first result of this section, which might be of some interest of in its own[19]. The lemma is based on a previous result of Alcantud and Rodríguez-Palmero (2002, Theorem 1) in the sense that the construction in the proof is very similar to theirs.

Lemma 1. *Let $\mathcal{D}^C : \mathbb{R}^2_{++} \to \mathbb{R}^2_+$ be an upper-hemicontinuous and convex valued correspondence such that for all $(x, y) \in \mathcal{D}^C(\omega, z)$, $\omega x + y = z$ and assume that $\mathcal{D}^C$ satisfies GARP (as in Definition 7). Then there exists a quasi-concave and monotone utility function u^C such that for all budgets (ω, z):*

$$\mathcal{D}^C(\omega, z) = \arg\max_{(x,y)} u^C(x, y) \ \text{ s.t. } \ \omega x + y \leq z.$$

[19]We would like to note that the theorem is easily generalizable to settings with more than two goods. For this, it suffices to adjust the definition of GARP (definition 7) by replacing the relation R by its transitive closure. A proof of this generalization is available on request from the authors.

The equality in the Lemma requires the demand correspondence to coincides with the set of all optimal utility maximising bundles for every possible budget $(\omega, z) \in \mathbb{R}^2_+$, this in fact being the demand correspondence generated by u^C.[20]

Proof. Assume that $\mathcal{D}^C$ is upper-hemicontinuous, convex valued and satisfies GARP. First, from Reny (2015), we have that GARP is sufficient for the existence of a utility function u^C which is quasi-concave and monotone and such that for all budgets (ω, z):

$$\mathcal{D}^C(\omega, z) \subseteq \underset{(x,y)}{\operatorname{argmax}}\, u^C(x, y) \text{ s.t. } \omega x + y \leq z. \tag{5}$$

We need to show that the reverse set inclusion also holds.

First let us list some facts:

- Condition (5) shows that if $(x, y) \in \mathcal{D}^C(\omega, z)$, then (x, y) is utility maximizing given the budget (ω, z). As such, from a simple revealed preference argument, we have that for all $(x, y) \in \mathcal{D}^C(\omega, z)$ and for all bundles (x', y'), if $z \geq \omega x' + y'$, then $u^C(x, y) \geq u^C(x', y')$.
- Next, using also monotonicity of u^C we have that for all $(x, y) \in \mathcal{D}^C(\omega, z)$ if $z > \omega x' + y'$ then $u^C(x, y) > u^C(x', y')$.

Notice that given Condition (5), we can finish the proof by showing that for any budget $(\omega, z) \in \mathbb{R}^2_{++}$, any $(x, y) \in \mathcal{D}^C(\omega, z)$ and any $(x', y') \notin \mathcal{D}^C(\omega, z)$ with $z \geq \omega x' + y'$, we have $u^C(x, y) > u^C(x', y')$.

If for this particular (x', y') we have the strict inequality $z > \omega x' + y'$, the previous bullet point immediately shows that $u^C(x, y) > u^C(x', y')$ and we are done. As such, we can focus on the case where $z = \omega x' + y'$.

So, let $(x', y') \notin D^C(\omega, z)$ and $z = \omega x' + y'$. Let $(\widetilde{x}, \widetilde{y})$ be the bundle in $\mathcal{D}^C(\omega, z)$ that is closest to (x', y').

$$(\widetilde{x}, \widetilde{y}) = \operatorname{argmin}_{(\bar{x},\bar{y}) \in \mathcal{D}^C(\omega,z)} \|(\bar{x}, \bar{y}) - (x', y')\|.$$

As $\mathcal{D}^C(\omega, z)$ is a bounded closed convex set and $(x', y') \notin \mathcal{D}^C(\omega, z)$, the bundle $(\widetilde{x}, \widetilde{y})$ is unique. As $(\widetilde{x}, \widetilde{y}) \in \mathcal{D}^C(\omega, z)$, we have that both (x, y) and $(\widetilde{x}, \widetilde{y})$ are utility maximizing, so $u^C(x, y) = u^C(\widetilde{x}, \widetilde{y})$. As such, it suffices to show that $u^C(\widetilde{x}, \widetilde{y}) > u^C(x', y')$.

Define $(\overline{x}, \overline{y}) = 0.5(\widetilde{x}, \widetilde{y}) + 0.5(x', y')$. Notice that $(\overline{x}, \overline{y}) \notin \mathcal{D}^C(\omega, z)$ by definition of $(\widetilde{x}, \widetilde{y})$. Now, define a sequence of prices:

$$\omega_i = \omega + \frac{1}{i}(\widetilde{x} - x').$$

and budgets

$$z_i = z + \frac{1}{2i}\left((\widetilde{x})^2 - (x')^2\right)$$

[20] Notice that the condition in Lemma 1 is a stronger statement than the condition that $\mathcal{D}^C(\omega, z) \subset \arg\max_{(x,y)} u^C(x, y)$ s.t. $\omega x + y \leq z$. The latter would not be sufficient to establish our result.

We have that:

$$\begin{aligned}\omega_i\widetilde{x}+\widetilde{y} &= \omega\widetilde{x}+\widetilde{y}+\frac{1}{i}(\widetilde{x}-x')\widetilde{x} = z+\frac{1}{i}(\widetilde{x})^2-\frac{1}{i}x'\widetilde{x},\\ &= z_i+\frac{1}{2i}((\widetilde{x})^2+(x')^2-2x'\widetilde{x}) = z_i+\frac{1}{2i}(\widetilde{x}-x')^2 > z_i.\end{aligned}$$

Also,

$$\omega_i\overline{x}+\overline{y} = \omega\overline{x}+\overline{y}+\frac{1}{2i}(\widetilde{x}-x')(\widetilde{x}+x') = z_i,$$

As for all $i\in\mathbb{N}$, $z_i = \dfrac{\omega_i\widetilde{x}+\widetilde{y}}{2}+\dfrac{\omega_i x'+y'}{2}$ and $z_i<\omega_i\widetilde{x}+\widetilde{y}$ we have that $z_i>\omega_i x'+y'$.

We have that ω_i converges to ω and z_i converges to z as i goes to infinity. As the set of budgets $\mathbb{R}^2_{++}$ is an open set, we have that $\omega_i, z_i>0$ for i large enough.

For all such large enough i take a bundle $(x_i,y_i)\in\mathcal{D}^C(\omega_i,z_i)$. As ω_i and z_i converge to ω and z respectively, by upper-hemicontinuity of $\mathcal{D}^C$, there should exist a subsequence $(x_{n_i},y_{n_i})_{i\in\mathbb{N}}$ that converges to an element $(\widehat{x},\widehat{y})\in\mathcal{D}^C(\omega,z)$.

Let $0<\varepsilon<\frac{|x'-\widetilde{x}|}{2}$. Let $D=\max\{1,\sup_{(x,y),(x',y')\in\mathcal{D}^C(\omega,z)}|x-x'|\}$ and pick $\delta>0$ such that $\delta<\min\left\{\frac{\varepsilon}{3},\frac{\varepsilon^2}{3D}\right\}$. Then, for i large enough $|\widehat{x}-x_{n_i}|<\delta$.

Notice that for all such i:

$$\begin{aligned}\omega x_{n_i}+y_{n_i} &= \omega_{n_i}x_{n_i}+y_{n_i}-\frac{1}{n_i}(\widetilde{x}-x')x_{n_i} = z_{n_i}-\frac{1}{n_i}(\widetilde{x}-x')x_{n_i},\\ &= z+\frac{1}{2n_i}\left[(\widetilde{x})^2+(x_{n_i})^2-2\widetilde{x}x_{n_i}-((x')^2-2x'x_{n_i}+(x_{n_i})^2)\right],\\ &= z+\frac{1}{2n_i}\left((\widetilde{x}-x_{n_i})^2-(x'-x_{n_i})^2\right)\end{aligned}$$

Let us show that $(\widetilde{x}-x_{n_i})^2-(x_{n_i}-x')^2<0$, so $\omega x_{n_i}+y_{n_i}<z$. If so, then $z_{n_i}>\omega_{n_i}x'+y'$ and $z>\omega x_{n_i}+y_{n_i}$. By a simple revealed preference argument, this implies that $u^C(\widetilde{x},\widetilde{y})>u^C(x_{n_i},y_{n_i})>u^C(x',y')$ which completes the proof.

Observe,

$$\begin{aligned}(x'-\widehat{x})^2 &= (x'-\widetilde{x})^2+(\widetilde{x}-\widehat{x})^2+2(x'-\widetilde{x})(\widetilde{x}-\widehat{x}),\\ &\geq (x'-\widetilde{x})^2+(\widetilde{x}-\widehat{x})^2 \geq 4\varepsilon^2+(\widetilde{x}-\widehat{x})^2\end{aligned}$$

The second lines follows from the fact that $\widetilde{x}$ is between x' and $\widehat{x}$, so $(x'-\widetilde{x})(\widetilde{x}-\widehat{x})\geq 0$. This last inequality implies that $|x'-\widehat{x}|>3\delta+|\widetilde{x}-\widehat{x}|$. If not, then:

$$\begin{aligned}(x'-\widehat{x})^2 &\leq (3\delta+|\widetilde{x}-\widehat{x}|)^2 = 9\delta^2+(\widetilde{x}-\widehat{x})^2+6\delta|\widetilde{x}-\widehat{x}|,\\ &\leq (\widetilde{x}-\widehat{x})^2+\varepsilon^2+6\delta D \leq (\widetilde{x}-\widehat{x})^2+\varepsilon^2+2\varepsilon^2,\\ &< (\widetilde{x}-\widehat{x})^2+4\varepsilon^2,\end{aligned}$$

Next,

$$|x' - \widehat{x}| \leq |x' - x_{n_i}| + |x_{n_i} - \widehat{x}| < |x' - x_{n_i}| + \delta.$$

and,

$$|\widetilde{x} - x_{n_i}| \leq |\widetilde{x} - \widehat{x}| + |\widehat{x} - x_{n_i}| < |\widetilde{x} - \widehat{x}| + \delta.$$

As such,

$$|x' - x_{n_i}| > |x' - \widehat{x}| - \delta > 2\delta + |\widetilde{x} - \widehat{x}| > |\widetilde{x} - x_{n_i}|$$

□

Lemma 1 allows us to replace requirement C) by the following:

C') the demand correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$ satisfy GARP, are convex valued and upper-hemicontinuous.

We now take up the challenge of constructing two upper-hemicontinuous, convex valued demand correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$ that satisfy A), B) and C'). We first explain the construction of these choice correspondences after which we will show that all the necessary properties are satisfied. Towards this end, we need to introduce some additional notation.

- First, we construct a number, r. For a bundle $(x, y) \in \mathbb{R}$ and a number δ, let $B((x, y), \delta)$ be the open ball with center (x, y) and radius δ. Let $r > 0$ be small enough such that for all $C, D \in \{A, B\}$ and all $t \in T^C, v \in T^D$ and all $(x, y) \in B((x_t^C, y_t^C), r)$ if,

$$z_v^D < \omega_v^D x_t^C + y_t^C \text{ then } z_v^D < \omega_v^D x + y,$$
$$z_v^D > \omega_v^D x_t^C + y_t^C \text{ then } z_v^D > \omega_v^D x + y.$$

  and such that for all $t \in T^C, v \in T^D$ and all $(x, y) \in B((x_v^D, y_v^D), r)$ if,

$$z_v^D < \omega_v^D x_t^C + y_t^C \text{ then } \omega_v^D x + y < \omega_v^D x_t^C + y_t^C,$$
$$z_v^D > \omega_v^D x_t^C + y_t^C \text{ then } \omega_v^D x + y > \omega_v^D x_t^C + y_t^C.$$

  We also take r to be small enough such that for all $C, D \in \{A, B\}$ and all $t \in T^C, v \in T^D$, if $(x_t^C, y_t^C) \neq (x_v^D, y_v^D)$ then $(x_t^C, y_t^C) \notin B((x_v^D, y_v^D), r)$ and if $y_t^C \neq 0$ then $y_t^C > r$.

- Next, for each $C \in \{A, B\}$, for all $t \in T^C$ and for all $\omega \leq \omega_t^C$, we define two numbers $\underline{y}_t^C(\omega)$ and $\overline{y}_t^C(\omega)$ in the following way.

$$\underline{y}_t^C(\omega) = y_t^C - r\frac{\omega_t^C - \omega}{\omega_t^C} \qquad \text{and} \qquad \overline{y}_t^C(\omega) = y_t^C + r\frac{\omega_t^C - \omega}{\omega_t^C}.$$

  Remark that $(x_t^C, \underline{y}_t^C(\omega)), (x_t^C, \overline{y}_t^C(\omega)) \in B((x_t^C, y_t^C), r)$. Given that r is quite small, both numbers are quite close to y_t^C. Notice that $\underline{y}_t^C(\omega)$ is increasing in ω and $\overline{y}_t^C(\omega)$ is decreasing in ω. Both agree with y_t^C if $\omega = \omega_t^C$. Finally, notice that $\overline{y}_t^C(\omega) > 0$ if $y_t^C > 0$ but $\overline{y}_t^C(\omega) < 0$ if $y_t^C = 0$.

- Finally, for all $\omega \leq \omega_t^C$, we define three income levels $\underline{z}_t^C(\omega), \overline{z}_t^C(\omega)$ and $\widehat{z}_t^C(\omega)$.

$$\begin{aligned}\underline{z}_t^C(\omega) &= \omega x_t^C + \underline{y}_t^C(\omega),\\ \overline{z}_t^C(\omega) &= \omega x_t^C + \overline{y}_t^C(\omega),\\ \widehat{z}_t^C(\omega) &= \omega_t^C x_t^C + \overline{y}_t^C(\omega).\end{aligned}$$

  $\underline{z}_t^C(\omega)$ is the budget necessary to buy $(x_t^C, \underline{y}_t^C)$ at price level ω. $\overline{z}_t^C(\omega)$ is the budget necessary to buy $(x_t^C, \overline{y}_t^C)$ at price level ω and $\widehat{z}_t^C(\omega)$ is the budget to buy $(x_t^C, \overline{y}_t^C)$ at price level ω_t^C. It is easily verified that, as $\underline{y}_t^C(\omega) \leq \overline{y}_t^C(\omega)$ and $\omega \leq \omega_t^C$,

$$\underline{z}_t^C(\omega) \leq \overline{z}_t^C(\omega) \leq \widehat{z}_t^C(\omega),$$

  where the inequalities are strict if $\omega < \omega_t^C$ and $x_t^C \neq 0$ and all three are equal when $\omega = \omega_t^C$.

For every $C \in \{A, B\}$ every $t \in T^C$ and every $\omega > 0$, we will now construct an "expansion path" $\Phi_t^C(\omega, z)$ in $\mathbb{R}_+^2$ (for $z > 0$). These will later be combined to construct the demand correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$. The idea behind the construction of $\Phi_t^C(\omega, z)$ is to make the consumption of x "as small as possible", while creating no revealed preference violation with the observed choice (x_t^C, y_t^C).

If $\omega > \omega_t^C$, we can simply put the bundles on the expansion path (x, y) equal to $(0, z)$.

Rule 1: if $\omega > \omega_t^C$, then

$$\Phi_t^C(\omega, z) = \{(0, z)\}.$$

Next if $\omega = \omega_t^C$, and if $z < z_t^C$ or $z > z_t^C$, we can again set $(x, y) = (0, z)$. On the other hand, if $z = z_t^C$ (so the two budgets are identical), we want to make sure that $(x_t^C, y_t^C) \in \Phi_t^C(\omega, z)$. We do this by simply equating it to all bundles on the line segment between $(0, z)$ and (x_t^C, y_t^C).

Rule 2: if $\omega = \omega_t^C$ and

2.1 if $z < z_t^C$ or $z > z_t^C$, then:

$$\Phi_t^C(\omega, z) = \{(0, z)\}.$$

2.2 if $z = z_t^C$, then:

$$\Phi_t^C(\omega, t) = \left\{(\alpha x_t^C, \alpha y_t^C + (1 - \alpha) z_t^C) | \alpha \in [0, 1]\right\},$$

Finally, we have the most tricky case where $\omega < \omega_t^C$. The construction for this case is illustrated in Figure 3.

If $z < \omega x_t^C + y_t^C$, it is not possible to have a revealed preference violation with observation $(x_t^C, y_t^C, \omega_t^C, z_t^C)$, so in principle, we could choose to put (x, y) equal to $(0, z)$. However, we will be a bit more conservative and only do this for $z < \underline{z}_t^C(\omega)$. This makes case 3.1 below.

Next for values of z above $\omega x_t^C + y_t^C$, it is possible to have a revealed preference violation. In order to avoid this, we need to make sure that for all $(x, y) \in \Phi_t^C(\omega, z)$, the value of x is not too small. In particular it should be to the right of the intersection between the budget (ω, z) and the budget (ω_t^C, z_t^C). Again, here we are going to be a bit more conservative. First, we are going to restrict ourselves to values of $z > \overline{z}_t^C(\omega)$ and $z \leq \widehat{z}_t^C(\omega)$. For these values, we equate (x, y) to the bundle on the intersection of the budget (ω, z) and the budget $(\omega_t^C, \widehat{z}_t^C(\omega))$. This makes case 3.4 below.

For values of z greater or equal to $\widehat{z}_t^C(\omega)$ the budgets (ω, z) and (ω_t^C, z_t^C) no longer intersect, so there can be no revealed preference violation. Therefore, we can put $(x, y) = (0, z)$. This makes case 3.5.

Next, we are left with an intermediate range where $z \in]\underline{z}_t^C(\omega), \overline{z}_t^C(\omega)]$. Here we need to distinguish between cases where $y_t^C > 0$ and $y_t^C = 0$. If $y_t^C > 0$, we simply pick $x = x_t^C$ and equate y equal to the value that makes sure that (x, y) is on the budget line (ω, z). This makes case 3.3.1. If $y_t^C = 0$, there is a range of values for z, for which it is impossible to set x equal to x_t^C, as this would imply that $y < 0$. For these cases we set y equal to zero and x equal to z/ω. This is summarized in 3.3.2.

To glue everything together we finally need to specify the value of $\Phi_t^C(\omega, z)$ when $z = \underline{z}_t^C(\omega)$. If $y_t^C > 0$, we define the correspondence to coincide with the line segment between $(0, \underline{z}_t^C(\omega))$ and $(x_t^C, \underline{y}_t^C)$. This is rule 3.2.1. if $y_t^C = 0$, we simply set $\Phi_t^C(\omega, z)$ to be the entire budget line defined by (ω, z). This is rule 3.2.2.

Rule 3: If $\omega < \omega_t^C$

3.1 if $z < \underline{z}_t^C(\omega)$, then: $\Phi_t^C(\omega, z) = \{(0, z)\}$.

3.2 if $z = \underline{z}_t^C(\omega)$,

3.2.1 and $y_t^C > 0$ then: $\Phi_t^C(\omega, t) = \left\{(\alpha x_t^C, \alpha \underline{y}_t^C(\omega) + (1-\alpha)\underline{z}_t^C(\omega)) | \alpha \in [0, 1]\right\}$.

This means that $\Phi_t^C(\omega, z)$ corresponds to the line segment on the budget (ω, z) to the left of $(x_t^C, \underline{y}_t^C(\omega))$.

3.2.2 and $y_t^C = 0$ then: $\Phi_t^C(\omega, t) = \{(x, y) | \omega x + y = z\}$.

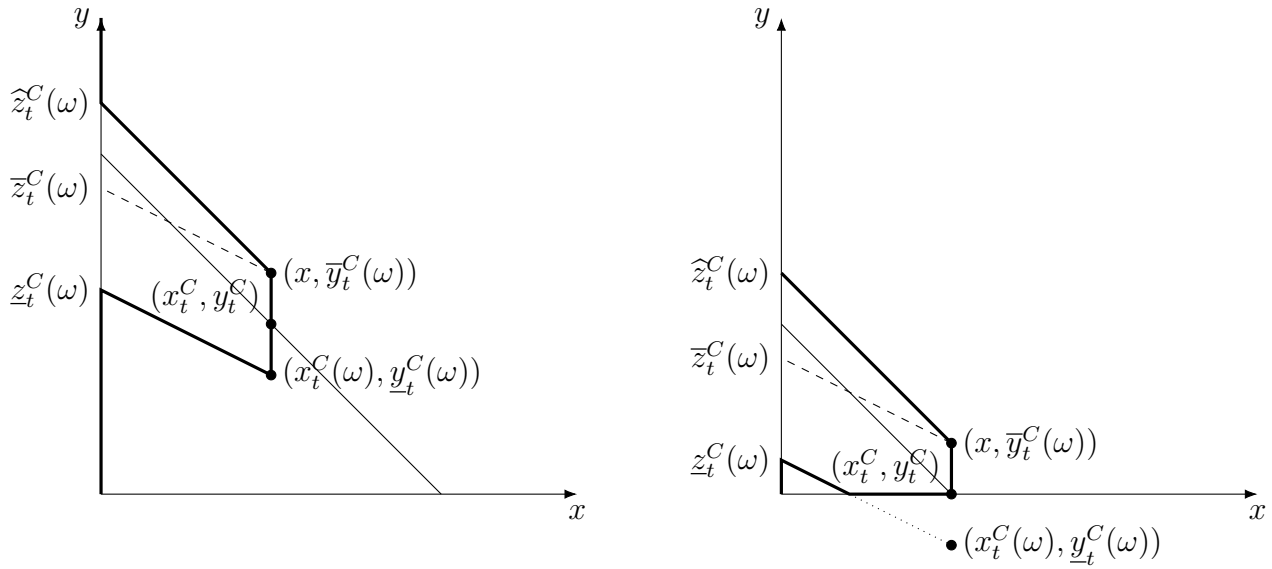
This means that $\Phi_t^C(\omega, z)$ corresponds to the entire budget line (ω, z).

3.3 if $z = \alpha \underline{z}_t^C + (1-\alpha)\overline{z}_t^C$ for some $\alpha \in]0, 1[$ (i.e., z is between $\underline{z}_t^C(\omega)$ and $\overline{z}_t^C(\omega)$) then

3.3.1 if $\alpha \underline{y}_t^C(\omega) + (1-\alpha)\overline{y}_t^C(\omega) \geq 0$, then: $\Phi_t^C(\omega, z) = \{(x_t^C, \alpha \underline{y}_t^C(\omega) + (1-\alpha)\overline{y}_t^C(\omega))\}$.

This means that $\Phi_t^C(\omega, z)$ contains the bundle on the intersection between the budget (ω, z) and the line segment between $(x_t^C, \underline{y}_t^C(\omega))$ and $(x_t^C, \overline{y}_t^C(\omega))$.

Figure 3: Construction of $\Phi_t^C(\omega, z)$ for $\omega < \omega_t^C$. Left picture for $y_t^C > 0$ and right picture when $y_t^C = 0$



3.3.2 if $\alpha \underline{y}_t^C(\omega) + (1-\alpha)\overline{y}_t^C(\omega) < 0$ then: $\Phi_t^C(\omega, z) = \{(z/\omega, 0)\}$.
Notice that this case can only occur if $y_t^C = 0$.

3.4 if $\widehat{z}_t^C(\omega) \geq z \geq \overline{z}_t^C(\omega)$, then: $\Phi_t^C(\omega, z) = \left\{ \left(\frac{\widehat{z}_t^C(\omega) - z}{\omega_t^C - \omega}, \frac{\omega_t^C z - \omega \widehat{z}_t^C(\omega)}{\omega_t^C - \omega} \right) \right\}$.
Here the bundle in $\Phi_t^C(\omega, z)$ is the unique bundle on the intersection of the budgets (ω, z) and $(\omega_t^C, \widehat{z}_t^C(\omega))$.

3.5 if $z > \widehat{z}_t^C(\omega)$, then: $\Phi_t^C(\omega, z) = \{(0, z)\}$.

We are almost done with the construction. We define:

$$\mathcal{D}^A(\omega, z) = \curlyvee \{\Phi_t^A(\omega, z) | t \in T^A\},$$
$$\mathcal{D}^B(\omega, z) = \curlyvee \{\Phi_t^C(\omega, z) | C \in \{A, B\}, t \in T^C\}.$$

Let us first establish that $\mathcal{D}^A$ and $\mathcal{D}^B$ are convex valued and upper-hemicontinuous (which is part of the requirement in C').

Lemma 2. *The correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$ are convex valued and upper-hemicontinuous.*

Proof. We will show this for $\mathcal{D}^B$. The proof for $\mathcal{D}^A$ is very similar.

Notice that for all $C \in \{A, B\}$ and $t \in T^C$, $\Phi_t^C(\omega, z)$ is a convex valued and upper-hemicontinous correspondence and that for all $(\omega, z) \in \mathbb{R}^2_{++}$, $\Phi_t^C(\omega, z) \subset \{(x, y) | \omega x + y = z\}$.

Now, let $\mathcal{A}(\omega, z)$ and $\mathcal{B}(\omega, z)$ be two convex valued, upper-hemicontinuous correspondences such that for all $(\omega, z) \in \mathbb{R}^2_{++}$, $\mathcal{A}(\omega, z)$ and $\mathcal{B}(\omega, z)$ are subsets of the budget line $\{(x, y) | \omega x + y = z\}$. Let us show that $\mathcal{A}(\omega, z) \curlyvee \mathcal{B}(\omega, z)$ is also convex valued and upper-hemicontinous. Once this is established, the proof follows by induction on the size of $T^A \cup T^B$.

To show convex-valuedness, let $(x, y), (x', y') \in \mathcal{A}(\omega, z) \curlyvee \mathcal{B}(\omega, z)$ and consider the bundle $(x_\alpha, y_\alpha) = (\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y')$ for some $\alpha \in [0, 1]$. Without loss of generality, assume that $x \leq x'$.

- If $(x, y) \in \mathcal{A}(\omega, z)$ and $(x', y') \in \mathcal{A}(\omega, z)$ then there exists $(\widetilde{x}, \widetilde{y}), (\widehat{x}, \widehat{y}) \in \mathcal{B}(\omega, z)$ such that $\widetilde{x} \leq x$ and $\widehat{x} \leq x'$. Without loss of generality, assume $\widetilde{x} \leq \widehat{x}$.

  As $\mathcal{A}(\omega, z)$ is convex, we have that $(x_\alpha, y_\alpha) \in \mathcal{A}(\omega, z)$. Then as $\widetilde{x} \leq x$ and $\widetilde{x} \leq \widehat{x} \leq x'$, we have that $\widetilde{x} \leq x_\alpha$ and as $(\widetilde{x}, \widetilde{y}) \in \mathcal{B}(\omega, z)$, we have that $(\widetilde{x}, \widetilde{y}) \curlyvee (x_\alpha, y_\alpha) = (x_\alpha, y_\alpha) \in \mathcal{A}(\omega, z) \curlyvee \mathcal{B}(\omega, z)$.

- If $(x, y) \in \mathcal{B}(\omega, z)$ and $(x', y') \in \mathcal{B}(\omega, z)$ we have a similar reasoning.

- Now assume that $(x, y) \in \mathcal{A}(\omega, z)$ and $(x', y') \in \mathcal{B}(\omega, z)$. Then there is a $(\widetilde{x}, \widetilde{y}) \in \mathcal{B}(\omega, z)$ such that $\widetilde{x} \leq x$.

  Then as $\widetilde{x} \leq x \leq x'$, we can find a number $\gamma \in [0, 1]$ such that $\gamma \widetilde{x} + (1-\gamma)x' = x_\alpha$. Notice that $\gamma \widetilde{y} + (1-\gamma)y' = y_\alpha$ as all bundles are on the same budget line. As $\mathcal{B}(\omega, z)$ is convex, we have that $(x_\alpha, y_\alpha) \in \mathcal{B}(\omega, z)$. As $x \leq x'$, we have that $x \leq x_\alpha$. As such, from $(x, y) \in \mathcal{A}(\omega, z)$ and $(x_\alpha, y_\alpha) \in \mathcal{B}(\omega, z)$, we have that $(x, y) \curlyvee (x_\alpha, y_\alpha) = (x_\alpha, y_\alpha) \in \mathcal{A}(\omega, z) \curlyvee \mathcal{B}(\omega, z)$.

- If $(x, y) \in \mathcal{B}(\omega, z)$ and $(x', y') \in \mathcal{A}(\omega, z)$, we have a similar reasoning.

To show upper-hemicontinuity, let $(\omega_i, z_i)_{i \in \mathbb{N}}$ be a convergent sequence of budgets in $\mathbb{R}^2_{++}$ with limit $(\omega, z) \in \mathbb{R}^2_{++}$ and let $(x_i, y_i)_{i \in \mathbb{N}}$ be a sequence of bundles such that for all $i \in \mathbb{N}$, $(x_i, y_i) \in \mathcal{A}(\omega_i, z_i) \curlyvee \mathcal{B}(\omega_i, z_i)$. It is clear that the sequence $(x_i, y_i)_{i \in \mathbb{N}}$ is bounded (as they are on the budgets (ω_i, z_i) which themselves converge to the budget (ω, z)). Also, if $(x_i, y_i)_{i \in \mathbb{N}}$ converges to, say (x, y) then either,

- there is a convergent subsequence $(x_{i_n}, y_{i_n})_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ $(x_{i_n}, y_{i_n}) \in \mathcal{A}(\omega_{i_n}, z_{i_n})$ and there are associated $(x'_{i_n}, y'_{i_n}) \in \mathcal{B}(\omega_{i_n}, z_{i_n})$ for which $x'_{i_n} \leq x_{i_n}$,

- or there is a subsequence $(x_{i_n}, y_{i_n})_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ $(x_{i_n}, y_{i_n}) \in \mathcal{B}(\omega_{i_n}, z_{i_n})$ and there are associated $(x'_{i_n}, y'_{i_n}) \in \mathcal{A}(\omega_{i_n}, z_{i_n})$ for which $x'_{i_n} \leq x_{i_n}$.

Without loss of generality, assume that the first is the case (the other case is treated by symmetry). As the sequence (x'_{n_i}, y'_{n_i}) is bounded, it has a further convergent subsequence, say with limit (x', y'). Then as $\mathcal{B}$ is upper-hemicontinuous $(x', y') \in \mathcal{B}(\omega, z)$. Also, along this subsequence, the sequence $(x_{i_n}, y_{i_n})_{n \in \mathbb{N}}$ still converges to (x, y) so, taking limits, we

find that $x \geq x'$. From upper-hemicontinuity of $\mathcal{A}$, also $(x, y) \in \mathcal{A}(\omega, z)$. This shows that $(x', y') \curlyvee (x, y) = (x, y) \in \mathcal{A}(\omega, z) \curlyvee \mathcal{B}(\omega, z)$, so upper-hemicontinuity is satisfied. □

The following shows that the correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$ satisfy the condition for Definition 1, i.e., condition A).

Lemma 3. *For all* $(\omega, z) \in \mathbb{R}^2_{++}$,

$$\mathcal{D}^B(\omega, z) = \mathcal{D}^A(\omega, z) \curlyvee \mathcal{D}^B(\omega, z) \qquad \textit{and} \qquad \mathcal{D}^A(\omega, z) = \mathcal{D}^A(\omega, z) \curlywedge \mathcal{D}^B(\omega, z).$$

Proof. Let $(x, y) \in \mathcal{D}^A(\omega, z)$ and let $(x', y') \in \mathcal{D}^B(\omega, z)$. If $(x', y') = (x', y') \curlyvee (x, y)$ (or equivalently $(x, y) = (x, y) \curlywedge (x', y')$), we are done. As such, assume that $x' < x$ (and $y' > y$).

Let us first show that $(x', y') \in \mathcal{D}^A(\omega, z)$. As $(x', y') \in \mathcal{D}^B(\omega, z)$ there should exist for all $t \in T^A$ and $v \in T^B$ bundles $(x_t, y_t) \in \Phi^A_t(\omega, z)$ and $(x_v, y_v) \in \Phi^B_v(\omega, z)$ such that $x_t \leq x'$ and $x_v \leq x'$. In particular, there should exist a $(\widetilde{x}, \widetilde{y}) \in \bigvee \{\Phi^A_t(\omega, z) | t \in T^A\} = \mathcal{D}^A_t(\omega, z)$ such that $\widetilde{x} \leq x'$. But then $\widetilde{x} \leq x' \leq x$. As $\mathcal{D}^A(\omega, z)$ is convex valued, we have that $(x', y') \in \mathcal{D}^A(\omega, z)$.

To see that $(x, y) \in \mathcal{D}^B(\omega, z)$, notice that as $(x, y) \in \mathcal{D}^A(\omega, z)$ and $(x', y') \in \mathcal{D}^B(\omega, z)$ there should exist for all $t \in T^A$ and $v \in T^B$, $(x_t, y_t) \in \Phi^A_t(\omega, z)$ such that $x_t \leq x (\leq x')$ and $(x_v, y_v) \in \Phi^B_v(\omega, z)$ such that $x_v \leq x'$. In addition, there is a $w \in T^A$ such that $(x, y) \in \Phi^A_w(\omega, z)$. But then, $(x, y) \in \bigvee \{\Phi^C_t(\omega, z) | C \in \{A, B\}, t \in T^C\}$.

□

To finish the proof, we are left to demonstrate that $\mathcal{D}^A$ and $\mathcal{D}^B$ satisfy GARP (rest of condition C') and that for all $C \in \{A, B\}$ and $t \in T^C$, $(x^C_t, y^C_t) \in \mathcal{D}^C(\omega^C_t, z^C_t)$ (condition B).

Let us first show the latter

Lemma 4. *For all* $C \in \{A, B\}$ *and all* $t \in T^C$, $(x^C_t, y^C_t) \in \mathcal{D}^C(\omega^C_t, z^C_t)$.

Proof. We show the proof for $C = B$ as the proof for $C = A$ is very similar.

First, notice that for all $v \in T^B$, $\overline{y}^B_v(\omega^B_v) = \underline{y}^B_v(\omega^B_v) = y^B_v$ and $\overline{z}^B_v(\omega^B_v) = \underline{z}^B_v(\omega^B_v) = \widehat{z}^B_v(\omega^B_v) = z^B_v$. As such:

$$(x^B_v, y^B_v) \in \Phi^B_v(\omega^B_v, z^B_v)$$

Now, towards a contradiction, assume that $(x^B_v, y^B_v) \notin \mathcal{D}^B(\omega^B_v, z^B_v)$. If so, then by construction of $\mathcal{D}^B$, there must be an individual $C \in \{A, B\}$ and an observation $t \in T^C$ such that for all $(x^*, y^*) \in \Phi^C_t(\omega^B_v, z^B_v)$):

$$x^* > x^B_v$$

There are several cases to consider, depending on the value of the budget (ω^B_v, z^B_v) and the value of ω^C_t.

1. If $\omega_v^B > \omega_t^C$ then by definition $\Phi_t^C(\omega_v^B, z_v^B) = \{(0, z_v^B)\}$, so $x^* = 0$, which gives a contradiction.

2. If $\omega_v^B = \omega_t^C$,

   2.1 and $z_v^B > z_t^C$ or $z_v^B < z_t^C$, then $\Phi_t^C(\omega_v^B, z_v^B) = \{(0, z_v^B)\}$, which gives the same contradiction: $x^* = 0$.

   2.2 and $z_v^B = z_t^C$ then (ω_t^C, z_t^C) and (ω_v^B, z_v^B) are identical. The definition gives that

   $$\Phi_t^C(\omega_v^B, z_v^B) = \{(\alpha x_t^C, \alpha y_t^C + (1-\alpha) z_t^C) | \alpha \in [0,1]\}.$$

   If we set $\alpha = 0$, we obtain that $(0, z_v^B) \in \Phi_t^C(\omega_v^B, z_v^B)$, which once more gives the desired contradiction: there is an $(x^*, y^*) \in \Phi_t^C(\omega_v^B, z_v^B)$ such that $x^* = 0$.

3. If $\omega_v^B < \omega_t^C$

   3.1 and $z_v^B < \underline{z}_t^C(\omega_v^B)$ then $\Phi_t^C(\omega_v^B, z_v^B) = \{(0, z_v^B)\}$, again a contradiction.

   3.2 and $z_v^B = \underline{z}_t^C(\omega_v^B)$ then,

   $$z_v^B = \underline{z}_t^C(\omega_v^B) = \omega_v^B x_t^C + \underline{y}_t^C(\omega_v^B) = \omega_v^B x_t^C + y_t^C - r \underbrace{\frac{\omega_t^C - \omega_v^B}{\omega_t^C}}_{>0} < \omega_v^B x_t^C + y_t^C$$

   As such, $z_v^B < \omega_v^B x_t^C + y_t^C$. By definition of r, this implies that $z_v^B < \omega_v^B x + y$ for all $(x, y) \in B((x_t^C, y_t^C), r)$, which contradicts with the fact that $(x_t^C, \underline{y}_t^C(\omega_v^B)) \in B((x_t^C, y_t^C), r)$ and $z_v^B = \underline{z}_t^C(\omega_v^B) = \omega_v^B x_t^C + \underline{y}_t^C(\omega_v^B)$.

   3.3 if $z_v^B = \alpha \underline{z}_t^C(\omega_v^B) + (1-\alpha)\overline{z}_t^C(\omega_t^B)$ for some $\alpha \in]0,1[$,

   - then if $z_v^B < \omega_v^B x_t^C + y_t^C$ we also have $z_v^B < \omega_v^B x_t^C + \underline{y}_t^C(\omega_v^B) = \underline{z}_t^C(\omega_v^B)$ as $(x_t^C, \underline{y}_t^C(\omega_v^B)) \in B((x_t^C, y_t^C), r)$ (by definition of r), a contradiction.
   - then, if $z_v^B > \omega_v^B x_t^C + y_t^C$ then $z_v^B > \omega_v^B x_t^C + \overline{y}_t^C(\omega_v^B) = \overline{z}_t^C(\omega_v^B)$ as $(x_t^C, \overline{y}_t^C(\omega_v^B)) \in B((x_t^C, y_t^C), r)$ (by definition of r), a contradiction.
   - conclude that $z_v^B = \omega_v^B x_t^C + y_t^C = z_t^C$. Then the unique element in $\Phi_t^C(\omega_v^B, z_v^B)$ equals (x_t^C, y_t^C). As such, by hypothesis $x^* = x_t^C > x_v^B$.
     But then,
     $$z_v^B = \omega_v^B x_t^C + y_t^C,$$
     and
     $$z_t^C = \omega_t^C x_t^C + y_t^C = \underbrace{\omega_v^B x_t^C + y_t^C}_{z_v^B} + (\omega_t^C - \omega_v^B) x_t^C,$$
     $$= z_v^B + \underbrace{(\omega_t^C - \omega_v^B)}_{>0} \underbrace{x_t^C}_{>x_v^B} > z_v^B + (\omega_t^C - \omega_v^B) x_v^B = \omega_t^C x_v^B + y_v^B.$$

If $C = B$, then $(x_v^B, y_v^B)\, R(x_t^B, y_t^B)$ and $(x_t^B, y_t^B)\, P(x_v^B, y_v^B)$ contradicting the assumption that $\mathcal{O}^B$ satisfies GARP. If $C = A$ then $(x_v^B, y_v^B)\, R(x_t^A, y_t^A)$ and $(x_t^A, y_t^A)\, P(x_v^B, y^B - v)$ and $\omega_v^B < \omega_t^A$, which contradicts the second part of condition ii of Theorem 2.

3.4 if $\widehat{z}_t^C(\omega_v^B) \geq z_v^B > \overline{z}_t^C(\omega_v^B)$, then

$$\Phi_t^C(\omega_v^B, z_v^B) = \left\{ \left(\frac{\widehat{z}_t^C(\omega_v^B) - z_v^B}{\omega_t^C - \omega_v^B}, \frac{\omega_t^C z_v^B - \omega_v^B \widehat{z}_t^C(\omega_v^B)}{\omega_t^C - \omega_v^B} \right) \right\}.$$

So,

$$(x^*, y^*) = \left(\frac{\widehat{z}_t^C(\omega_v^B) - z_v^B}{\omega_t^C - \omega_v^B}, \frac{\omega_t^C z_v^B - \omega_v^B \widehat{z}_t^C(\omega_v^B)}{\omega_t^C - \omega_v^B} \right)$$

In particular, (x^*, y^*) is the unique bundle on the intersection of the budget (ω_v^B, z_v^B) and the budget $(\omega_t^C, \widehat{z}_t^C(\omega_v^B))$

By assumption,

$$z_v^B > \overline{z}_t^C(\omega_v^B) = \omega_v^B x_t^C + \overline{y}_t^C(\omega_v^B) = \omega_v^B x_t^C + y_t^C + r \underbrace{\frac{\omega_t^C - \omega_v^B}{\omega_t^C}}_{>0} > \omega_v^B x_t^C + y_t^C.$$

So if the conditions in part ii of Theorem 2 are satisfied, then

$$z_t^C < \omega_t^C x_v^B + y_v^B.$$

as $(x_t^C, \overline{y}_t^C(\omega_v^B)) \in B((x_t^C, y_t^C), r)$ and $z_t^C < \omega_t^C x_v^B + y_v^B$, we have that by definition of r that $\omega_t^C x_t^C + \overline{y}_t^C(\omega_v^B) < \omega_t^C x_v^B + y_v^B$. As such,

$$\omega_t^C x^* + y^* = \widehat{z}_t^C(\omega_v^B) = \omega_t^C x_t^C + \overline{y}_t^C(\omega_v^B) < \omega_t^C x_v^B + y_v^B.$$

So,

$$\begin{aligned} &\omega_t^C x^* + \underbrace{y^*}_{=z_v^B - \omega_v^B x^*} < \omega_t^C x_v^B + \underbrace{y_v^B}_{=z_v^B - \omega_v^B x_v^B}, \\ \leftrightarrow &z_v^B + (\omega_t^C - \omega_v^B) x^* < z_v^B + (\omega_t^C - \omega_v^B) x_v^B, \\ \rightarrow &0 < (\omega_t^C - \omega_v^B)(x_v^B - x^*). \end{aligned}$$

As the first term on the right hand side is strictly positive, we must have that $x_v^B > x^*$, which contradicts the hypothesis.

3.5 if $z_v^B > \widehat{z}_t^C(\omega_v^B)$ then $\Phi_t^C(\omega_v^B, z_v^B) = \{(0, z_v^B)\}$ so $x^* = 0$ which gives the desired contradiction.

□

Finally, we need to show that the demand correspondences $\mathcal{D}^A$ and $\mathcal{D}^B$ satisfy GARP as in Definition 7 (remaining part of C'). We will do this for $\mathcal{D}^B$ as the proof for $\mathcal{D}^A$ is very similar. The following lemmata will allow us to shorten the proofs of the various cases.

Lemma 5. *Let (ω, z) and (ω', z') be two budgets and let (x, y) and (x', y') be two bundles on these respective budgets. Then if,*

- $(x, y) \in \Phi_t^C(\omega, z)$,
- $\widehat{z}_t^C(\omega) \geq z \geq \underline{z}_t^C(\omega)$,
- $\omega_t^C \geq \omega > \omega'$,
- $z \geq \omega x' + y'$ *and* $z' \geq \omega' x + y$,

then $\widehat{z}_t^C(\omega') > z' > \underline{z}_t^C(\omega')$.

Proof. Assume the conditions of the Lemma hold. Then,

$$z \geq \omega x' + \underbrace{y'}_{=z'-\omega' x'} = z' + \underbrace{(\omega - \omega')}_{>0} x' \geq z'.$$

As such,

$$\widehat{z}_t^C(\omega') = \omega_t^C x_t^C + \overline{y}_t^C(\omega') = \omega_t^C x_t^C + y_t^C + \underbrace{r \frac{\omega_t^C - \omega'}{\omega_t^C}}_{> \frac{\omega_t^C - \omega}{\omega_t^C}},$$

$$> \omega_t^C x_t^C + \underbrace{y_t^C + r \frac{\omega_t^C - \omega}{\omega_t^C}}_{=\overline{y}_t^C(\omega)} = \omega_t^C x_t^C + \overline{y}_t^C(\omega) = \widehat{z}_t^C(\omega) \geq z \geq z'.$$

This establishes one inequality.

For the other one, we need to consider several cases.

- First consider $\underline{z}_t^C(\omega) \leq z \leq \overline{z}_t^C(\omega)$. As $(x, y) \in \Phi_t^C(\omega, z)$, notice that $x \leq x_t^C$.

So,

$$\begin{aligned}
\underline{z}_t^C(\omega') &= \omega' x_t^C + \underline{y}_t^C(\omega') = \omega' x_t^C + y_t^C - r \underbrace{\frac{\omega_t^C - \omega'}{\omega_t^C}}_{> \frac{\omega_t^C - \omega}{\omega_t^C}}, \\
&< \omega' x_t^C + \underbrace{y_t^C - r\frac{\omega_t^C - \omega}{\omega_t^C}}_{=\underline{y}_t^C(\omega)} = \omega' x_t^C + \underline{y_t^C}(\omega), \\
&= \underbrace{\omega x_t^C + \underline{y}_t^C(\omega)}_{\underline{z}_t^C(\omega)} + (\omega' - \omega) x_t^C = \underline{z}_t^C(\omega) + \underbrace{(\omega' - \omega)}_{<0} x_t^C, \\
&\leq z + (\omega' - \omega)x = \omega' x + y \leq z'.
\end{aligned}$$

- If $\overline{z}_t^C(\omega) < z \leq \widehat{z}_t^C(\omega)$, notice first that $\omega \neq \omega_t^C$, as an equality would mean that $\overline{z}_t^C(\omega) = \widehat{z}_t^C(\omega)$. As such, we can assume that $\omega < \omega_t^C$. In this case, we have $x = \frac{\widehat{z}_t^C(\omega) - z}{\omega_t^C - \omega}$ and $y = \frac{\omega_t^C z - \omega \widehat{z}_t^C(\omega)}{\omega_t^C - \omega}$.

  Let us show that $z' \geq \omega' x_t^C + \overline{y}_t^C(\omega)$. If not then,

$$\begin{aligned}
&\omega' x_t^C + \overline{y}_t^C(\omega) > z' \geq \omega' x + y, \\
\rightarrow &(\omega' - \omega_t^C) x_t^C + \underbrace{\omega_t^C x_t^C + \overline{y}_t^C(\omega)}_{=\widehat{z}_t^C(\omega)} > \omega' x + y, \\
\rightarrow &(\omega' - \omega_t^C) x_t^C + \underbrace{\omega_t^C x + y}_{=\widehat{z}_t^C(\omega)} > \omega' x + y, \\
\rightarrow &(\omega_t^C - \omega')(x_t^C - x) < 0.
\end{aligned}$$

  A contradiction as $(\omega_t^C > \omega')$ and $x_t^C \geq x$.

  Then,

$$\begin{aligned}
\underline{z}_t^C(\omega') &= \omega' x_t^C + \underline{y}_t^C(\omega'), \\
&= \omega' x_t^C + y_t^C - r \underbrace{\frac{\omega_t^C - \omega'}{\omega_t^C}}_{> \frac{\omega_t^C - \omega}{\omega_t^C}} < \omega' x_t^C + y_t^C - r\frac{\omega_t^C - \omega}{\omega_t^C}, \\
&\leq \omega' x_t^C + y_t^C + r\frac{\omega_t^C - \omega}{\omega_t^C} = \omega' x_t^C + \overline{y}_t^C(\omega) \leq z'.
\end{aligned}$$

□

Lemma 6. *For any two budgets (ω, z) and (ω', z') and any two bundles (x, y) and (x', y') from these budgets (i.e., $\omega x + y = z$ and $\omega' x' + y' = z'$), if:*

- $(x, y) \in \Phi_t^C(\omega, z)$,
- $x' \geq x^*$ *for some* $(x^*, y^*) \in \Phi_t^C(\omega', z')$
- $\widehat{z}_t^C(\omega) \geq z \geq \underline{z}_t^C(\omega)$,
- $\widehat{z}_t^C(\omega') > z' > \underline{z}_t^C(\omega')$,
- $\omega_t^C \geq \omega > \omega'$,
- $z \geq \omega x' + y'$ *and* $z' \geq \omega' x + y$,

then $(x, y) = (x', y')$.

Proof. If $x = x'$ or $y = y'$ then the conclusion easily follows from the last inequalities. Therefore, from now on, we assume that $x \neq x'$ and $y \neq y'$.

Notice that, using the budget constraints, the two revealed preference conditions can be rewritten as:

$$z \geq z' + (\omega - \omega')x' \text{ and } z' \geq z + (\omega' - \omega)x,$$

Adding up these two inequalities gives $0 \geq (\omega - \omega')(x' - x)$. As $\omega > \omega'$ we have that $x^* \leq x' < x$, which will be useful (the last strict inequality follows from the fact that $x \neq x'$).

Notice that as as $\widehat{z}_t^C(\omega') > z > \underline{z}_t^C(\omega')$, we have that for all $(x^*, y^*) \in \Phi_t^C(\omega', z')$, $x^* > 0$. As such $x > x^* > 0$ which rules out cases 1), 2.1), 3.1) and 3.5) for determining $\Phi_t^C(\omega, z)$. The fact that $\widehat{z}_t^C(\omega') > z > \underline{z}_t^C(\omega')$ and $\omega_t^C > \omega'$ rules out cases 1), 2.1), 2.2), 3.1) and 3.5) for determining $\Phi_t^C(\omega', z')$.

Also, notice that as $(x, y) \in \Phi_t^C(\omega, z)$ then $x \leq x_t^C$. As such, we have,

$$x^* \leq x' < x \leq x_t^C,$$

This means that, as $(x^*, y^*) \in \Phi_t^C(\omega', z')$, the set $\Phi_t^C(\omega', z')$ can also not be determined by rules 3.3.1).

To summarize, we only need to check cases 2.2), 3.2.1), 3.2.2), 3.3.2) and 3.4) for $\Phi_t^C(\omega, z)$ and case 3.3.2) and 3.4) for $\Phi_t^C(\omega', z')$.

If $\Phi_t^C(\omega', z')$ is governed by 3.3.2) then $y_t^C = 0$ so $(x^*, y^*) = (z'/\omega', 0)$. Then as $x' \geq x^*$ and (x', y') is on the budget (z', ω') we have that also $x' = z'/\omega'$ and $y' = 0$ this gives:

$$z' \geq \omega' x + y > \omega' x' + y = z' + y,$$

which is impossible. So we are left with case 3.4) for $\Phi_t^C(\omega', z')$. As such, we have that $(x^*, y^*) = \left(\frac{\widehat{z}_t^C(\omega') - z'}{\omega_t^C - \omega'}, \frac{\omega_t^C z' - \omega' \widehat{z}_t^C(\omega')}{\omega_t^C - \omega'}\right)$. In particular, the bundle (x^*, y^*) is the unique bundle on both budgets $(\omega_t^C, \widehat{z}_t^C(\omega'))$ and (ω', z').

We distinguish the following cases for $(x, y) \in \Phi_t^C(\omega, z)$.

- If $\underline{z}_t^C(\omega) \leq z \leq \overline{z}_t^C(\omega)$. Then,

$$\begin{aligned}
z \leq \overline{z}_t^C(\omega) &= \omega x_t^C + \overline{y}_t^C(\omega) = \omega x_t^C + y_t^C + r \underbrace{\frac{\omega_t^C - \omega}{\omega_t^C}}_{<\frac{\omega_t^C - \omega'}{\omega_t^C}}, \\
&< \omega x_t^C + y_t^C + r\frac{\omega_t^C - \omega'}{\omega_t^C} = \omega x_t^C + \overline{y}_t^C(\omega'), \\
&= (\omega - \omega_t^C)x_t^C + \underbrace{\omega_t^C x_t^C + \overline{y}_t^C(\omega')}_{=\widehat{z}_t^C(\omega')} = \underbrace{(\omega - \omega_t^C)}_{<0}\underbrace{x_t^C}_{>x^*} + \underbrace{\omega_t^C x^* + y^*}_{=\widehat{z}_t^C(\omega')}, \\
&< (\omega - \omega_t^C)x^* + \omega_t^C x^* + y^* = \omega x^* + y^*,
\end{aligned}$$

  So,

$$z < \omega x^* + \underbrace{y^*}_{=z' - \omega' x^*} = z' + (\omega - \omega')x^*,$$

  Together with $z' \leq z + (\omega' - \omega)x$ (obtained from $z \geq z' + (\omega - \omega')x$), we obtain $0 < (\omega - \omega')(x^* - x)$. From $\omega > \omega'$, we therefore obtain that $x^* > x > x'$, a contradiction.

- if $\widehat{z}_t^C(\omega) \geq z > \overline{z}_t^C(\omega)$. First, notice that $\omega < \omega_t^C$ as $\omega = \omega_t^C$ gives $\widehat{z}_t^C(\omega) = \overline{z}_t^C(\omega)$, a contradiction. We have that now (x, y) is the unique bundle on the intersection of the budgets (ω, z) and $(\omega_t^C, \widehat{z}_t^C(\omega))$

$$\begin{aligned}
z = \omega x + y &= (\omega - \omega_t^C)x + \underbrace{\omega_t^C x + y}_{=\widehat{z}_t^C(\omega)}, \\
&= \underbrace{(\omega - \omega_t^C)}_{<0}\underbrace{x}_{>x^*} + \underbrace{\omega_t^C x_t^C + \overline{y}_t^C(\omega)}_{=\widehat{z}_t^C(\omega)}, \\
&< (\omega - \omega_t^C)x^* + \omega_t^C x_t^C + y_t^C + r \underbrace{\frac{\omega_t^C - \omega}{\omega_t^C}}_{<\frac{\omega_t^C - \omega'}{\omega_t^C}}, \\
&< (\omega - \omega_t^C)x^* + \omega_t^C x_t^C + \underbrace{y_t^C + r\frac{\omega_t^C - \omega'}{\omega_t^C}}_{=\overline{y}_t^C(\omega')}, \\
&= (\omega - \omega_t^C)x^* + \underbrace{\omega_t^C x_t^C + \overline{y}_t^C(\omega')}_{=\widehat{z}_t^C(\omega')}, \\
&= (\omega - \omega_t^C)x^* + \underbrace{\omega_t^C x^* + y^*}_{=\widehat{z}_t^C(\omega')} = \omega x^* + y^*,
\end{aligned}$$

  We obtain a similar contradiction as in the previous case.

□

We are now ready to show that $\mathcal{D}^B(\omega, z)$ satisfies GARP. Assume that $(x, y) \in \mathcal{D}^B(\omega, z)$ and $(x', y') \in \mathcal{D}^B(\omega', z')$ with $z \geq \omega x' + y'$ (i.e., $(x, y)\, R^B(x', y')$) and $z' \geq \omega' x + y$ (i.e., $(x', y')\, R^B(x, y)$). To demonstrate GARP, it suffices to show that both inequalities are in fact equalities.

Let us first exclude some easy cases

- If $\omega = \omega'$ then immediately, we have that $z \geq \omega' x' + y' = z'$ and $z' \geq \omega x + y = z$. So $z = z'$, and the two budgets are identical. In this case, both inequalities are indeed equalities.

  From now on, we restrict ourselves to the cases where $\omega \neq \omega'$.

- Next, if $x = x'$ then also immediately $z \geq \omega x + y' = z - y + y'$ which implies that $y \geq y'$. Next, $z' \geq \omega' x' + y = z' - y' + y$ which implies that $y' \geq y$. This shows that also $y = y'$. As $(x, y) = (x', y')$, the two inequalities are again equalities. A similar reasoning holds if $y = y'$.

  From now on, we can therefore restrict ourselves to cases where $x \neq x'$ and $y \neq y'$.

Notice that if $x = 0$ (and therefore $x' > 0$), then $y = z$, so we get the conditions,

$$z' \geq z \text{ and } z \geq z' + (\omega - \omega')x',$$

Combining the two gives:

$$0 \geq (\omega - \omega')x',$$

which means that $\omega' > \omega$ (as $\omega' \neq \omega$ and $x' > 0$).

Now, it is time to wrap things up. Without loss of generality, assume that $\omega' < \omega$. Also, from the definition of $\mathcal{D}^B$ there exist $C \in \{A, B\}$ and $t \in T^C$ such that $(x, y) \in \Phi_t^C(\omega, z)$.

- If $(x, y) \in \Phi^C(\omega, z)$ is determined by case 1), 2.1), 3.1) or 3.5) then $x = 0$, so by the reasoning above, $\omega < \omega'$. However, by assumption $\omega' < \omega$, a contradiction.

- This means that $\underline{z}_t^C(\omega) \leq z \leq \widehat{z}_t^C(\omega)$ and $\omega_t^C \geq \omega (> \omega')$. Notice that all conditions of Lemma 5 are satisfied. As such, we can conclude that $\underline{z}_t^C(\omega') < z' < \widehat{z}_t^C(\omega')$.

  By assumption $(x', y') \in \mathcal{D}^B(\omega', z')$, which means that there exists a bundle $(x^*, y^*) \in \Phi_t^C(\omega')$ such that $x^* \leq x'$. Conclude that all conditions of Lemma 6 are satisfied. This implies that $x = x'$.

□

A.3 Proof of Theorem 3

Fix a budget $(\omega, z) \in \mathbb{R}^2_{++}$. All statements and results in this proof are conditional on this budget. The proof of Theorem 3 is based on two lemmata.

Lemma 7. *For all $\bar{u}$, the set of bundles $(x, y) \in \mathbb{R}^2_+$ on the budget line (i.e., for which $\omega x + y = z$) and for which $u^B(x, y) \geq \overline{u}$ is a closed convex set.*

Proof. Follows from quasi-concavity of B and continuity of u^B. □

Let us denote by $\mathcal{F}(\bar{u})$ the set of all bundles $(x, y) \in \mathbb{R}^2_+$ that satisfy $\omega x + y = z$ and such that $u^B(x, y) \geq \overline{u}$.

For $\mathcal{F}(\bar{u})$ nonempty, let $\underline{x}(\bar{u}) = \min\{x|(x, y) \in \mathcal{F}(\bar{u})\}$, $\overline{x}(\bar{u}) = \max\{x|(x, y) \in \mathcal{F}(\bar{u})\}$ and similarly, $\underline{y}(\bar{u}) = \min\{y|(x, y) \in \mathcal{F}(\bar{u})\}$ and $\overline{y}(\bar{u}) = \max\{y|(x, y) \in \mathcal{F}(\bar{u})\}$. As $\mathcal{F}(\bar{u})$ is compact, these exist.

As upper contour sets are nested, it is also easy to see that if $\bar{u}' > \bar{u}$ and if $\mathcal{F}(\bar{u}')$ is non-empty then:

$$\underline{x}(\bar{u}) \leq \underline{x}(\bar{u}') \leq \overline{x}(\bar{u}') \leq \overline{x}(\bar{u}) \text{ and } \underline{y}(\bar{u}) \leq \underline{y}(\bar{u}') \leq \overline{y}(\bar{u}') \leq \overline{y}(\bar{u}),$$

Lemma 8. *For all $\bar{u}$*

1. *If $\overline{x}(\bar{u}) = \underline{x}(\bar{u})$ then $\omega \in \Omega^B(\overline{x}(\bar{u}), \overline{y}(\bar{u}))$.*
2. *If $\overline{x}(\bar{u}) > \underline{x}(\bar{u})$ and if $\omega^B \in \Omega^B(\overline{x}(\bar{u}), \overline{y}(\bar{u}))$ then $\omega^B \leq \omega$.*
3. *If $(x, y) \in \mathbb{R}^2_+$ satisfies $\omega x + y = z$ if $x > \overline{x}(\bar{u})$ and $\omega^B \in \Omega^B(x, y)$ then $\omega^B < \omega$.*

Proof. For the first, assume that $\overline{x}(\bar{u}) = \underline{x}(\bar{u})$. In this case, there is a unique point on the budget line, namely $(\overline{x}(\bar{u}), \overline{y}(\bar{u}))$, that reaches the utility level $u^B(\overline{x}(\bar{u}), \overline{y}(\bar{u})) = \overline{u}$. As such, for all $(x, y) \in \mathbb{R}^2_+$: $\omega\overline{x}(\bar{u}) + \overline{y}(\bar{u}) \geq \omega x + y$ implies $u^B(\overline{x}(\bar{u}), \overline{y}(\bar{u})) \geq u^B(x, y)$. This means that $\omega \in \Omega^B(\overline{x}(\bar{u}), \overline{y}(\bar{u}))$.

For the second, let $\overline{x}(\bar{u}) > \underline{x}(\bar{u})$, let $\omega^B \in \Omega^B(\overline{x}(\bar{u}), \overline{y}(\bar{u}))$, and assume, towards a contradiction that $\omega^B > \omega$. Now,

$$\begin{aligned} 0 = \omega(\overline{x}(\bar{u}) - \underline{x}(\bar{u})) + \overline{y}(\bar{u}) - \underline{y}(\bar{u}) < \omega^B(\overline{x}(\bar{u}) - \underline{x}(\bar{u})) + \overline{y}(\bar{u}) - \underline{y}(\bar{u}) \\ \rightarrow \omega^B\overline{x}(\bar{u}) + \overline{y}(\bar{u}) > \omega^B\underline{x}(\bar{u}) + \underline{y}(\bar{u}). \end{aligned}$$

From this, it follows that $u^B(\overline{x}(\bar{u}), \overline{y}(\bar{u})) > u^B(\underline{x}(\bar{u}), \underline{y}(\bar{u}))$. This, however contradicts the assumption that both are equal to $\overline{u}$.

For the third part assume that $\omega x + y = z$ and $x > \overline{x}(\bar{u})$. Assume that $\omega^B \in MRS^B(x, y)$ and, towards a contradiction that $\omega^B \geq \omega$. Then

$$\begin{aligned} 0 = \omega(x - \overline{x}(\bar{u})) + y - \overline{y}(\bar{u}) \leq \omega^B(x - \overline{x}(\bar{u})) + y - \overline{y}(\bar{u}) \\ \rightarrow \omega^B x + y \geq \omega^B\overline{x}(\bar{u}) + \overline{y}(\bar{u}). \end{aligned}$$

As such, $u^B(x,y) \geq u^B(\overline{x}(\bar{u}), \overline{y}(\bar{u})) = \overline{u}$. This contradicts the fact that $\overline{x}(\bar{u})$ was defined as the highest value of x for some $(x,y) \in \mathbb{R}^2_+$ on the budget line for which $u^B(x,y) \geq \overline{u}$. $\square$

Now, for the proof of the main theorem. Let us first show that $\mathcal{D}(\omega, z, \bar{u}') \curlyvee \mathcal{D}(\omega, z, \bar{u}) \subseteq \mathcal{D}(\omega, z, \bar{u}')$.

Let $(x,y) \in \mathcal{D}(\omega, z, \bar{u})$ and $(x',y') \in \mathcal{D}(\omega, z, \bar{u}')$. If $x' \geq x$, we are done as $(x,y) \curlyvee (x',y') = (x',y') \in \mathcal{D}(\omega, z, \bar{u}')$. As such, assume that $x' < x$. We distinguish between several cases. Note that $(x,y) \in \mathcal{F}(\bar{u})$ and $(x',y') \in \mathcal{F}(\bar{u}')$.

- Case $x \leq \underline{x}(\bar{u}')$. In this case, $x' < x \leq \underline{x}(\bar{u}')$, a contradiction with $(x',y') \in \mathcal{F}(\bar{u}')$.
- Case $\underline{x}(\bar{u}') \leq x \leq \overline{x}(\bar{u}')$. In this case, as $\mathcal{F}(\bar{u}') \subseteq \mathcal{F}(\bar{u})$, both (x,y) and (x',y') were feasible for both problems as $(x,y), (x',y') \in \mathcal{F}(\bar{u}')$. This means that $u^A(x,y) = u^A(x',y')$ so $(x,y) \in \mathcal{D}(\omega, z, \bar{u}')$, as we wanted to show.
- Case $\underline{x}(\bar{u}') < \overline{x}(\bar{u}') < x$. Take a value $\tilde{x}$ such that $\overline{x}(\bar{u}') < \tilde{x} < x$ and let $\tilde{y} = z - \omega\tilde{x}$.

  Let $\omega^B \in \Omega^B(\overline{x}(\bar{u}'), \overline{y}(\bar{u}'))$. By Lemma 8, we know that $\omega^B \leq \omega$. By the single crossing condition, we can find an $\omega^A \in \Omega^A(\overline{x}(\bar{u}'), \overline{y}(\bar{u}'))$ such that $\omega^A \leq \omega^B \leq \omega$. As such,

  $$\begin{aligned} 0 = \omega(\tilde{x} - \overline{x}(\bar{u}')) + \tilde{y} - \overline{y}(\bar{u}') &\geq \omega^A(\tilde{x} - \overline{x}(\bar{u}')) + \tilde{y} - \overline{y}(\bar{u}'). \\ \rightarrow \omega^A\overline{x}(\bar{u}') + \overline{y}(\bar{u}') &\geq \omega^A\tilde{x} + \tilde{y}. \end{aligned}$$

  This shows that $u^A(\overline{x}(\bar{u}'), \overline{y}(\bar{u}')) \geq u^A(\tilde{x}, \tilde{y})$.

  Using Lemma 8 once more, we know that for all $\widetilde{\omega}^B \in \Omega^B(\tilde{x}, \tilde{y})$, $\widetilde{\omega}^B < \omega$. Then pick $\widetilde{\omega}^A \in \Omega^A(\tilde{x}, \tilde{y})$ such that $\widetilde{\omega}^A \leq \widetilde{\omega}^B < \omega$. Then

  $$\begin{aligned} 0 = \omega(x - \tilde{x}) + y - \tilde{y} &> \widetilde{\omega}^A(x - \tilde{x}) + y - \tilde{y}, \\ \rightarrow \widetilde{\omega}^A\tilde{x} + \tilde{y} &> \widetilde{\omega}^A x + y. \end{aligned}$$

  This shows that $u^A(\tilde{x}, \tilde{y}) > u^A(x,y)$. By transitivity $u^A(\overline{x}(\bar{u}'), \overline{y}(\bar{u}')) > u^A(x,y)$. But this contradicts the fact that $(\overline{x}(\bar{u}'), \overline{y}(\bar{u}'))$ is on the budget line, and $(\underline{x}(\bar{u}'), \underline{y}(\bar{u}')) \in \mathcal{F}(\bar{u})$ together with $(x,y) \in \mathcal{D}(\omega, z, \bar{u})$.
- Case $\underline{x}(\bar{u}') = \overline{x}(\bar{u}') < x$.

  From the lemma, we know that in this case $\omega \in \Omega^B(\overline{x}(\bar{u}'), \overline{y}(\bar{u}'))$. So there is a $\omega^A \in \Omega^A(\overline{x}(\bar{u}'), \overline{y}(\bar{u}'))$ such that $\omega^A \leq \omega$. The remainder of the proof follows the argument in the preceding case.

Next, let us show that $\mathcal{D}(\omega, z, \bar{u}') \subseteq \mathcal{D}(\omega, z, \bar{u}) \curlyvee \mathcal{D}(\omega, z, \bar{u}')$. Let $(x',y') \in \mathcal{D}(\omega, z, \bar{u}')$. If there is an $(x,y) \in \mathcal{D}(\omega, z, \bar{u})$ such that $x \leq x'$ we are done. So, towards a contradiction, assume that for all $(x,y) \in \mathcal{D}(\omega, z, \bar{u})$, $x > x'$. We go over the 4 cases as before.

- Case $x \leq \underline{x}(\bar{u}')$. In this case $x' < x \leq \underline{x}(\bar{u}')$, so $x' \notin \mathcal{F}(\bar{u}')$ which is impossible.
- Case $\underline{x}(\bar{u}') \leq x \leq \overline{x}(\bar{u}')$. In this case $x, x' \in \mathcal{F}(\bar{u}') \subseteq \mathcal{F}(\bar{u})$. As such, $u^A(x,y) = u^A(x',y')$. This means that $(x',y') \in \mathcal{D}(\omega, z, \bar{u})$, a contradiction.
- Case $\underline{x}(\bar{u}') < \overline{x}(\bar{u}') < x$. This gives the same contradiction as the third case in the first part of the proof.
- Case $\underline{x}(\bar{u}') = \overline{x}(\bar{u}') < x$. This gives the same contradiction as the last case in the first part of the proof.

A.4 Proof of Theorem 4

Consider (ω, z) and (ω', z') such that $\omega' \leq \omega$ and consider $(x,y) \in \mathcal{D}(\omega, z, \bar{u})$ and $(x',y') \in \mathcal{D}(\omega', z', \bar{u}')$. Let us show the first condition. The second is almost entirely analogues. As $\bar{u}' > \bar{u}$, Theorem 3 shows there is a $(\tilde{x}, \tilde{y}) \in \mathcal{D}(\omega, z, \bar{u}')$ such that $\tilde{x} \geq x$ (and $\tilde{y} \leq y$).

From $(x,y)\, R(x',y')$, we obtain $\omega\tilde{x} + \tilde{y} = \omega x + y \geq \omega x' + y'$. As such $(\tilde{x}, \tilde{y})\, R(x',y')$. Given that $u^B(x',y') \geq \overline{u}'$ and $u^B(\tilde{x}, \tilde{y}) \geq \overline{u}'$, it follows from a simple revealed preference argument that:

$$u^A(\tilde{x}, \tilde{y}) \geq u^A(x',y').$$

On the other hand using $(x',y')\, P(x,y)$, and $\omega' \leq \omega$, we get that:

$$\omega'\tilde{x} + \tilde{y} = \underbrace{(\omega' - \omega)}_{\leq 0} \underbrace{\tilde{x}}_{\geq x} + z \leq (\omega' - \omega)x + z = \omega' x + y < \omega' x' + y'.$$

As such, $(x',y')\, P(\tilde{x}, \tilde{y})$. As $u^B(x',y') \geq \overline{u}'$ and $u^B(\tilde{x}, \tilde{y}) \geq \overline{u}'$, we obtain $u^A(x',y') > u^A(\tilde{x}, \tilde{y})$, a contradiction.

B Additional results for illustrative example

B.1 X-GARP conditional on GARP

In this appendix we measure how stringent the X-GARP condition is for individuals who behave in a way consistent with utility maximization (i.e., satisfy GARP) both when interacting with a stranger and with a parent.

Table 2 reports the proportion of individuals satisfying X-GARP among all of those that satisfy GARP separately when playing with a parent and with a stranger. To measure the power of this condition, we start by generating random datasets consistent with GARP using a Monte Carlo Markov Chain (MCMC) approach. Next, we compute the share of those datasets that reject X-GARP. The last row of Table 2 shows that 43.7 percent of such simulated datasets reject X-GARP.

Table 2: Pass rates and power for Theorem 2

Setting	Variable	X-GARP (conditional on GARP)
$S \succeq_x P$	Pass rate	0.892
$P \succeq_x S$	Pass rate	0.424
	Power	0.437

B.2 Treatment groups

Tables 3 reports the pass rates by treatment group as defined in the original paper of Porter and Adams (2015). Treatment groups were defined based to the level of information parents had about the game played in the experiment.

Table 3: Pass rates for Theorem 2

Group	Setting	Pass rate		
		GARP	X-GARP	GARP & X-GARP
1	$S \succeq_x P$	0.864	0.803	0.727
	$P \succeq_x S$	0.864	0.424	0.394
2	$S \succeq_x P$	0.850	0.883	0.767
	$P \succeq_x S$	0.850	0.367	0.333
3	$S \succeq_x P$	0.781	0.891	0.734
	$P \succeq_x S$	0.781	0.359	0.328

C Additional results for PROGRESA data

Table 4: Balance table

	Treatment		*Control*			t-test
	Mean	Sd	Mean	Sd	Diff.	p-value
Members' characteristics						
Head male	0.99	0.08	0.99	0.08	0.000	0.989
Head indigenous	0.38	0.48	0.38	0.49	0.002	0.916
Head age	36.95	10.58	37.01	10.23	0.053	0.892
Head educ.	2.29	0.92	2.26	0.95	-0.037	0.298
Spouse educ.	2.20	0.93	2.19	0.96	-0.015	0.675
N. children						
Young	2.33	1.33	2.29	1.38	-0.031	0.554
Older	1.29	1.33	1.31	1.36	0.014	0.786
Expenditure						
Food (all)	794.17	362.93	743.81	351.05	-50.364	0.000
h food	263.69	207.02	239.76	210.57	-23.935	0.003
ℓ food	530.48	247.10	504.05	246.68	-26.429	0.006
Ratio h/l food	0.54	0.42	0.53	0.50	-0.009	0.604
Price index						
h food	10.29	2.10	10.27	2.37	-0.015	0.811
ℓ food	5.90	1.11	5.95	1.18	0.046	0.307
N. obs.	1892		1111		3003	