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Bounding risk aversion

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Abstract

We propose a revealed preference method to non-parametrically bound the coefficients of relative (and absolute) risk aversion in an expected utility framework. Our approach abstains from placing functional form restrictions on the Bernoulli utility function. Our method is applicable to any finite number of observations on choices over Arrow-Debreu contingent claims, and can be efficiently implemented using linear or quadratic programming techniques. We illustrate our results using a large-scaled experimental data set.

Keywords: Expected Utility, Revealed Preference, Risk Aversion.

JEL-codes: D11, C14, D81

1 Introduction

Expected utility is the standard theory of choice under uncertainty. A central notion in this theory concerns the degree of risk aversion. This paper presents a revealed preference framework to compute the tightest (uniform) bounds on the Arrow-Pratt measures of relative (and absolute) risk aversion in an expected utility setting with Arrow-Debreu

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contingent commodities. Our procedure is purely nonparametric in the sense that we abstain from placing any functional form restriction on the Bernoulli utility function aside from standard smoothness and regularity conditions like strict monotonicity and concavity. Our methods are developed within a revealed preference context, and consequently, assumes only knowledge of state probabilities, prices and chosen quantities of the commodities for a finite set of budgets. We generalize our results by showing how our method can be used to bound the degree of risk aversion in the subjective expected utility model, in which case we assume that only prices and quantities are observed. Finally, we show how the method can easily be modified to provide tightest bounds on the degree of risk aversion over some restricted range of income and how we can take into account errors, like perception bias with respect to prices.

Most of the literature that deals with estimating the degree of risk aversion, assumes that utility functions take on a particular functional form.[1] The most common functional forms are the Constant Absolute Risk Aversion (CARA) and Constant Relative Risk Aversion (CRRA) families. In general, however, there is no reason to assume that individuals have a constant degree of relative or absolute risk aversion over all levels of income. When this assumption fails, there is no unique coefficient of risk aversion and these coefficients can be bounded at best.

We derive a revealed preference procedure that is able to compute the tightest uniform bound on the relative and absolute risk aversion values for a dataset which is consistent with the Expected Utility (EU) maximisation model. The width of this bound allows us to draw inference concerning the minimal variation in risk aversion that is necessary in order to explain the observed choices. Interestingly, if this width is zero, then the individual's choices can effectively be rationalized by a CRRA or CARA utility function. On the other hand, if the width is strictly positive then the behaviour is rationalizable only if the risk aversion is allowed to vary with the level of income. In this sense, the width of the bounds can also be seen as a measure for the minimal necessary deviation from the constant risk aversion hypothesis.

We provide an illustration by computing bounds on the relative risk aversion using data

[1]See Barseghyan, Molinari, O'Donogue, and Teitelbaum (2018) for an overview.

from a large-scaled experimental data sets collected by Choi, Kariv, Müller, and Silverman (2014). For the vast majority of subjects, the resulting bounds are tight. In addition, for many subjects the data are consistent with an upper bound on relative risk aversion below 2. Taken together, these findings indicate that the bounds are highly informative for a large share of the sample.

Literature overview Closest in spirit to our paper are Varian (1988) and Chetty (2006). Within an expected utility framework, Varian (1988) uses revealed preference methods to place bounds on the measures of risk aversion. Varian's methods are applicable to data sets consisting of only a single observation. In contrast, our methods are applicable to any finite number of observations, and are based on solving simple linear programming problems, which implies that they can be efficiently solvable in polynomial time. Most importantly, however, while Varian's methods require additional assumptions on the Bernoulli utility functions, such as assuming either increasing or decreasing risk aversion, our method avoids imposing such additional assumptions.

Chetty (2006) develops a method to bound the risk aversion using data on labour supply. His method requires an initial estimate of the elasticity between labour supply and consumption. Our approach is different in the sense that it applies to settings in which choices over Arrow–Debreu contingent claims are observed.[2] Although not a straightforward exercise, our methods can, in principle, be extended to labor supply data.

Our paper is also related to the literature that provides revealed preference characterizations of the expected utility model. Varian (1983) and Green and Srivastava (1986) proposed the first such characterization (See also Diewert (2012)). More recently, Kubler, Selden, and Wei (2014), Echenique and Saito (2015) and Polisson, Quah, and Renou (2020) proposed simple axiomatic conditions for a data set of Arrow-Debreu contingent claims to be rationalized by the (subjective) expected utility model. Finally, Heufer

[2]It should be well noted that the Arrow-Debreu setup is much more general than it may appear at first sight. Indeed, if we instead of choices over basic Arrow-Debreu securities would observe choices over more general assets, then it is always possible to transform this into an observationally equivalent context where investors choose over (fictional) Arrow-Debreu securities. This follows from the work of Ross (1978), Breeden and Litzenberger (1978) and Varian (1987).

(2014) develops a general method to compare risk aversion of different investors based on revealed preference methods. This method allows to draw ordinal conclusions if one investor is more risk averse than another investor (sometimes it gives inconclusive comparisons), but the method does not give estimates (or bounds) on individual risk aversion.

The next section introduces the setup and the expected utility model. Section 3 contains our main result. Section 4 contain extensions and adaptations. Section 5 contains the empirical application, and Section 6 concludes. The Appendix contains all proofs.

2 The expected utility model

We consider a setting where an expected utility maximizing decision maker (DM) chooses an optimal allocation of state contingent commodities. Let S be a finite set of states. We assume that state $s \leq S$ occurs with objective probability $\pi_s \in (0,1)$ with $\sum_{s \in S} \pi_s = 1$.[3] For every state s, the DM can purchase an Arrow-Debreu state contingent asset which pays an amount of one if state s occurs and zero otherwise. We denote by $x_s \in \mathbb{R}_+$ the amount of this state contingent commodity, and denote its price by $p_s \in \mathbb{R}_{++}$. The DM has Bernoulli utility function u and chooses the allocation $(x_s)_{s \leq S}$ in order to optimize expected utility $\sum_{s \leq S} \pi_s u(x_s)$ subject to a linear budget constraint.

Our analysis starts with a finite data set $\mathcal{D} = (p_{t,s}, \pi_{t,s}, x_{t,s})_{t \leq T, s \leq S}$ of state contingent prices $p_{t,s} > 0$, probabilities $\pi_{t,s} > 0$ and consumption quantities $x_{t,s}$ for a finite number of observations $t \leq T$ over a finite number of states $s \leq S$.

We use the following definition of expected utility rationalizability.

Definition 1. *The dataset $\mathcal{D} = (p_{t,s}, \pi_{t,s}, x_{t,s})_{t \leq T, s \leq S}$ is said to be Expected Utility (EU) rationalizable with Bernoulli utility function $u : \mathbb{R}_+ \to \mathbb{R}$ if, for all observations $t \leq T$:*

$$(x_{t,s})_{s \leq S} \in \arg\max_{(x_s)_{s \leq S}} \sum_{s \leq S} \pi_{t,s} u(x_s) \quad \textit{subject to} \quad \sum_{s \leq S} p_{t,s} x_s \leq \sum_{s \leq S} p_{t,s} x_{t,s}. \tag{1}$$

Expected utility rationalizability requires the observed choices to maximize the expected utility over all feasible allocations.

[3] We discuss the extension to subjective probabilities in Section 4.2.

If u is concave, C^1, and strictly increasing, then the first order conditions for a solution to (1) are given by:

$$u'(x_{t,s}) = \lambda_t \frac{p_{t,s}}{\pi_{t,s}}, \text{ if } x_{t,s} > 0 \tag{2}$$

$$u'(x_{t,s}) \leq \lambda_t \frac{p_{t,s}}{\pi_{t,s}}, \text{ if } x_{t,s} = 0 \tag{3}$$

where $\lambda_t > 0$ is the Lagrange multiplier corresponding to the budget constraint and $u'(x_{t,s})$ is the derivative of u at $x_{t,s}$. Note that if u is concave and C^1, then the marginal utility function, u', is non-increasing. Hence,

$$x_{t,s} \leq x_{v,k} \text{ implies } u'(x_{v,k}) \leq u'(x_{t,s}).$$

Using the first order conditions (2)–(3) this gives:

$$x_{t,s} \leq x_{v,k} \text{ and } x_{v,k} \neq 0 \text{ implies } \lambda_v \frac{p_{v,k}}{\pi_{v,k}} \leq \lambda_t \frac{p_{t,s}}{\pi_{t,s}}$$

Taking natural logs of both sides and using the notation $\delta_t = \ln(\lambda_t)$ yields that:

$$x_{t,s} \leq x_{v,k} \text{ and } x_{v,k} \neq 0 \text{ implies } \delta_v + \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right) \leq \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right). \tag{4}$$

As shown above, the existence of values $(\delta_t)_{t \leq T}$ that satisfy (4) is clearly necessary for a data set $\mathcal{D}$ to be EU rationalizable by a concave, C^1 and strictly increasing Bernoulli utility function. The following shows that it is also sufficient.

Theorem 2. *A dataset $\mathcal{D} = (p_{t,s}, \pi_{t,s}, x_{t,s})_{t \leq T, s \leq S}$ is EU rationalizable by some concave, strictly increasing and C^1 Bernoulli utility function if and only if there exist values $(\delta_t)_{t \leq T}$ such that (4) is satisfied for all observations $t, v \leq T$ and states $s, k \leq S$.*

3 Relative risk aversion

The Arrow-Pratt measure of relative risk aversion for an income level $x \in \mathbb{R}_{++}$ is given by:

$$R(x) = -\frac{u''(x)}{u'(x)} x.$$

Note that this measure (being the income-elasticity of marginal utility) only makes sense for Bernoulli utility functions that are C^2 and for values of $x > 0$. As such, we will assume in this section that for our dataset $\mathcal{D}$, all observed consumption amounts $x_{t,s}$ are strictly positive.[4] For the measure of absolute risk aversion, this assumption can be relaxed.

Define:

$$R^L = \inf_{x \in \mathbb{R}_{++}} R(x) \qquad \text{and} \qquad R^U = \sup_{x \in \mathbb{R}_{++}} R(x).$$

Our main goal is to place tight bounds on the difference $R^U - R^L$. In particular, we are looking for a well behaved Bernoulli utility function u that explains the observed data in the sense of Definition 1 and for which the difference $R^U - R^L$ is as small as possible. In a sense, this implies that we are looking for a utility function u that explains the observed data and is as close as possible to a CRRA utility function.

Towards this end, suppose that $\mathcal{D}$ is EU rationalizable by a C^2, strictly increasing and concave Bernoulli utility function u, and note that the measure of relative risk aversion can be expressed as $R(x) = -\,\partial \ln(u'(x))/\partial \ln(x)$. Then for all observations $t, v \leq T$ and all states $s, k \leq S$ with $x_{t,s} \leq x_{v,k}$ we have that:

$$\begin{aligned}
\ln(u'(x_{t,s})) - \ln(u'(x_{v,k})) &= -\int_{\ln(x_{t,s})}^{\ln(x_{v,k})} \frac{\partial \ln(u'(x))}{\partial \ln(x)} \mathrm{d}\ln(x),\\
&= \int_{\ln(x_{t,s})}^{\ln(x_{v,k})} -\frac{\partial \ln(u'(x))}{\partial \ln(x)} \mathrm{d}\ln(x),\\
&= \int_{\ln(x_{t,s})}^{\ln(x_{v,k})} R(x)\, \mathrm{d}\ln(\mathrm{x}).
\end{aligned}$$

Since, by definition, $R(x) \leq R^U$, and the latter is independent of x, we obtain:

$$\begin{aligned}
\ln(u'(x_{t,s})) - \ln(u'(x_{v,k})) &\leq \int_{\ln(x_{t,s})}^{\ln(x_{v,k})} R^U \mathrm{d}\ln(x)\\
&= R^U[\ln(x_{v,k}) - \ln(x_{t,s})].
\end{aligned}$$

Substituting out the first order conditions (2) and denoting $\delta_t = \ln(\lambda_t)$ gives:

$$x_{t,s} \leq x_{v,k} \text{ implies } \left[\delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right)\right] - \left[\delta_v + \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right)\right] \leq R^U\left(\ln\left(x_{v,k}\right) - \ln\left(x_{t,s}\right)\right). \quad (5)$$

[4] If $x_{t,s} = 0$ for some observation t and state s, then equation (3) requires $u'(0)$ to be bounded. This in turn gives a value of $R(0) = 0$. In this case $\inf_{x \in \mathbb{R}_+} R(x) = 0$, which gives the (rather uninteresting) case where the lower bound on the relative risk aversion equals zero.

Equation (5) expresses the fact that the log difference in marginal utility between $x_{t,s}$ and $x_{v,s}$ should be no larger than $R^U(\ln(x_{v,k}) - \ln(x_{t,s}))$. Indeed, if this property would not be satisfied, there should be at least one value x between $x_{t,s}$ and $x_{v,k}$ such that $-\,\partial \ln(u'(x))/\partial \ln(x) > R^U$, a contradiction.

By an analogous argument, using $R(x) \geq R^L$, we obtain that for all observations $t, v \leq T$ and all states $s, k \leq S$:

$$x_{t,s} \leq x_{v,k} \text{ implies } \ln(u'(x_{t,s})) - \ln(u'(x_{v,k})) \geq R_U[\ln(x_{v,k}) - \ln(x_{t,s})].$$

If, again, we substitute out the marginal level of utilities using (2), we obtain the following set of linear inequalities:

$$x_{t,s} \leq x_{v,k} \text{ implies } \left[\delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right)\right] - \left[\delta_v - \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right)\right] \geq R^L\left(\ln\left(x_{v,k}\right) - \ln\left(x_{t,s}\right)\right). \quad (6)$$

The intuition is similar as the one for condition (5). Note that, for $R^L \geq 0$, condition (6) always implies condition (4).

To estimate the bounds on the relative risk aversion, we proceed by replacing in equations (5) and (6) the values of R^U and R^L by the unknowns ρ^U and ρ^L. Then we look for values of these unknowns that have a minimal difference.

OP.I: $\min_{\rho^U, \rho^L \geq 0, (\delta_t)_{t \in T}} \rho^U - \rho^L$ subject to

$$\left[\delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right)\right] - \left[\delta_v + \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right)\right] \leq \rho^U\left(\ln\left(x_{v,k}\right) - \ln\left(x_{t,s}\right)\right),$$
$$\left[\delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right)\right] - \left[\delta_v + \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right)\right] \leq \rho^L\left(\ln\left(x_{v,k}\right) - \ln\left(x_{t,s}\right)\right),$$

for all $x_{t,s} \leq x_{v,k}$

Given that condition (6) is stronger than condition (4), feasibility of **OP.I** also requires EU rationalizability of the dataset $\mathcal{D}$. In fact, as soon as (6) is satisfied for some value of R^L, **OP.I** will be feasible for some values of ρ^U and ρ^L.[5] We will discuss possible solutions for cases where $\mathcal{D}$ is not EU rationalizable in subsection 4.4.

Our next main result shows that the solution to **OP.I.** effectively provides the tightest bound on the range of relative risk aversion levels.

[5]Setting $R_L = 0$ and R_U high enough will do the job.

Theorem 3. *Consider a data set $\mathcal{D} = (p_{t,s}, \pi_{t,s}, x_{t,s})_{t \leq T, s \leq S}$.*

(a) If $\mathcal{D}$ is EU rationalizable with a strictly increasing, concave and C^2 Bernoulli utility function, $u : \mathbb{R}_{++} \to \mathbb{R}$, with risk aversion bounds R^U and R^L, then **OP.I** *has a feasible solution and its optimal values, ρ^U, ρ^L, satisfy $\rho^U - \rho^L \leq R^U - R^L$.*

(b) If **OP.I** *has an optimal solution ρ^U, ρ^L, then for all $\varepsilon > 0$, $\mathcal{D}$ is rationalizable with a strictly increasing, concave, C^∞ Bernoulli utility function u with risk aversion bounds R^U and R^L that satisfy $R^L = \rho^L$ and $R^U \leq \rho^U + \varepsilon$.*

(c) If **OP.I** *has an optimal solution ρ^U, ρ^L with $\rho^L > 0$, then for all $0 < \varepsilon < \rho^L$, $\mathcal{D}$ is rationalizable with a strictly increasing, concave, C^∞ Bernoulli utility function u with risk aversion bounds R^L and R^U that satisfy $R^L \geq \rho^L - \varepsilon$ and $R^U = \rho^U$.*

Theorem 3 show that the optimal solution to program **OP.I** gives tight bounds on the difference between the risk aversion bounds R^U and R^L. The first part of the theorem states that the estimated gap $\rho^U - \rho^L$ is always a lower bound on the true gap $R^U - R^L$. The second part of Theorem 3 shows that these estimates are arbitrarily tight. In particular, we can always find a utility function that rationalizes the dataset and such that the bounds on the risk aversion R^U and R^L from this utility function are approximated aribtrarily close by the estimates ρ^U and ρ^L.

Note that the latter two statements of Theorem 3 rely on the (arbitrarily small) parameter ε. The reason for stating the theorem in this way is somewhat technical and due to the requirement that the underlying utility function is required to be C^2 (which, in turn, is necessary for the relative risk aversion to be well defined). The following example illustrates why we need to set $\varepsilon > 0$.

Example 1. *Consider a dataset with a single observation t and three states s, k, ℓ that gives the following values:*

$$\begin{cases} \ln(x_{t,s}) = 1, \\ \ln(x_{t,k}) = 2, \\ \ln(x_{t,\ell}) = 3, \end{cases} \qquad \textit{and} \quad \begin{cases} \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) = 4, \\ \ln\left(\frac{p_{t,k}}{\pi_{t,k}}\right) = 3.5 \\ \ln\left(\frac{p_{t,\ell}}{\pi_{t,\ell}}\right) = 2 \end{cases}$$

*One easily checks that the constraints in **OP.I** give the requirements that $\rho^L \leq 0.5$ and $1 \leq \rho^U$. As such, the solution of **OP.I** provides us with optimal values of $\rho^L = 0.5$ and $\rho^U = 1$.*

Let u be a C^2 utility function that rationalizes this data. Let us show that it is impossible that $R^U = \rho^U$ and $R^L = \rho^L$. Towards a contradiction, assume that $R^U = \rho^U$ and $R^L = \rho^L$. Figure 1 gives an illustration. Points (a), (b) and (c) plot the value of the log marginal utility $\ln(u'(x_{t,s})), \ln(u'(x_{t,k}))$ and $\ln(u'(x_{t,\ell}))$ against the three values $\ln(x_{t,s})$, $\ln(x_{t,k})$ and $\ln(x_{t,\ell})$. The first order conditions for EU rationality tell us that:

$$\ln(u'(x_{t,s})) - \ln(u'(x_{t,k})) = \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) - \ln\left(\frac{p_{t,k}}{\pi_{t,k}}\right) = 0.5,$$
$$\ln(u'(x_{t,k})) - \ln(u'(x_{t,\ell})) = \ln\left(\frac{p_{t,k}}{\pi_{t,k}}\right) - \ln\left(\frac{p_{t,\ell}}{\pi_{t,\ell}}\right) = 1,$$

This means that the vertical distance between the points (a) and (b) must be 0.5, while the vertical distance between (b) and (c) equals 1. Now, if we were to plot the values of $\ln(u'(x))$ against $\ln(x)$ for all values of $x > 0$, we should therefore obtain a curve that:

i. passes through the points (a), (b) and (c),

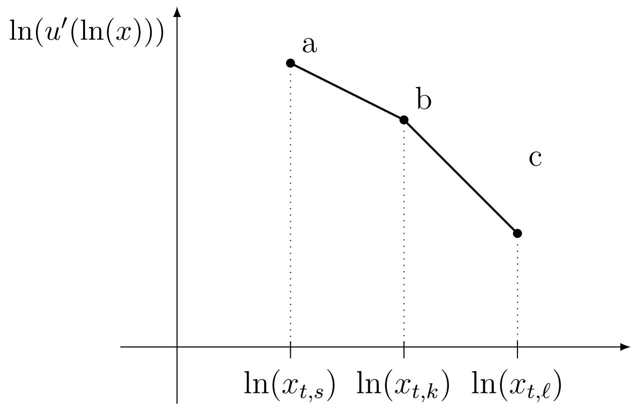
ii. is C^1 (as we assumed u to be C^2),

iii. whose slope equals $-R(x)$, which has absolute value contained in the interval $[R^L, R^U] = [0.5, 1]$

The first and last condition together imply that this curve coincides with the black line depicted in Figure 1. However this curve cannot be C^1 as it has a kink at the point (b). This shows that that dataset is not rationalizable by a C^2 utility function u for which $R(x) \in [\rho^L, \rho^U] = [0.5, 1]$. On the other hand, for any $\varepsilon > 0$ it is possible to obtain a C^∞ curve interpolating (a), (b) and (c), whose slope has absolute value in the interval $[\rho^L - \varepsilon, \rho^U] = [0.5 - \varepsilon, 1]$ or $[\rho^L, \rho^U + \varepsilon] = [0.5, 1 + \varepsilon]$.

Note that program **OP.I** is linear in all unknowns, and consequently, can be efficiently implemented (i.e., in polynomial time) using simple linear programming methods. Further

Figure 1: Plot of log consumption levels against log marginal utility levels



notice that, although there appear to be $2TS(TS-1)$ linear inequalities to be imposed, we only need to impose the inequalities (5) and (6) for consecutive values of $x_{t,s}$ and $x_{v,k}$.[6] This reduces (in general) the number of inequalities to $2(TS-1)$.

When the solution to **OP.I** gives optimal values $\rho^L = \rho^U$, then Theorem 3 shows that the data set $\mathcal{D}$ can be rationalized by a utility function that comes arbitrarily close to a CRRA Bernoulli utility function. In fact the following stronger condition holds in this case.

Corollary 1. *A dataset $\mathcal{D} = (p_{t,s}, \pi_{t,s}, x_{t,s})_{t\leq T, s\leq S}$ is rationalizable by a CRRA utility function if and only if the program **OP.I** has optimal value* 0*, i.e.* $\rho^U = \rho^L$.

Thus, $\rho^U = \rho^L$ in the program **OP.I** is a necessary and sufficient condition for the data set to be rationalizable by the EU-model with a constant relative risk aversion Bernoulli function.

4 Extensions

In this section, we present several extensions to our result. First, we discuss the case of estimating bounds on the measure of Absolute Risk Aversion (ARA). Next, we show how our model can easily be adjusted when the EU framework is generalized to a framework

[6] The quantities $x_{t,s} \leq x_{v,k}$ are consecutive if and only if there is no other observation $w \leq T$ and state $\ell \leq S$ such that $x_{t,s} < x_{w,\ell} < x_{v,k}$.

of Subjective Expected Utility maximization (SEU). We also show how we can modify the framework to estimate bounds on risk aversion over some income ranges. Finally, we show how to account for possible measurement or optimization errors.

4.1 Absolute risk aversion

The Arrow-Pratt measure of absolute risk aversion at the income level x is defined in the following way:

$$A(x) = -\frac{u''(x)}{u'(x)} = -\frac{\partial \ln(u'(x))}{\partial x}.$$

Similarly to the measure of relative risk aversion, we can define uniform bounds on the measure of absolute risk aversion:

$$A^L = \inf_{x \in \mathbb{R}_+} A(x) \text{ and } A^U = \sup_{x \in \mathbb{R}_+} A(x).$$

Following the steps in Section 3, but now for the coefficient of absolute risk aversion, we can derive that:

$$x_{t,s} \leq x_{v,k} \text{ implies } \begin{cases} \ln(u'(x_{t,s})) - \ln(u'(x_{v,k})) \leq A^u(x_{v,k} - x_{t,s}), \\ \ln(u'(x_{t,s})) - \ln(u'(x_{v,k})) \geq A^L(x_{v,k} - x_{t,s}) \end{cases}.$$

Hence, substituting the first order condition (2), taking logs and defining $\delta_t = \ln(\lambda_t)$ and $\delta_v = \ln(\lambda_v)$, we obtain the condition:

$$x_{t,s} \leq x_{v,k} \text{ implies } \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) - \delta_v - \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) \leq A^U\left(x_{v,k} - x_{t,s}\right). \tag{7}$$

and:

$$x_{t,s} \leq x_{v,k} \text{ implies } \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) - \delta_v - \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right) \geq A^L(x_{v,k} - x_{t,s}). \tag{8}$$

In order to calculate bounds on the coefficient of absolute risk aversion, it suffices to replace the conditions (5) and (6) with (7) and (8) in the linear programs **OP.I**. This gives:

OP.II: $\min_{\rho^U, \rho^L \geq 0, (\delta_t)_{t \in T}} \rho^U - \rho^L$ subject to Eqs. (7) and (8) where we replace A^U with ρ^U and A^L with ρ^L.

We have the following two results which are analogous to Theorem 3 and Corollary 1.

Theorem 4. *Consider a data set $\mathcal{D} = (p_{t,s}, \pi_{t,s}, x_{t,s})_{t \leq T, s \leq S}$.*

(a) If $\mathcal{D}$ is rationalizable with a strictly increasing, concave and C^2 Bernoulli utility function, $u : \mathbb{R}_{++} \to \mathbb{R}$ with risk aversion bounds A^U and A^L, then **OP.II** *has a feasible solution and its optimal values, ρ^U, ρ^L, satisfy $\rho^U - \rho^L \leq A^U - A^L$.*

(b) If **OP.II** *has an optimal solution ρ^L, ρ^U then for all $\varepsilon > 0$, $\mathcal{D}$ is rationalizable with a strictly increasing, concave, C^∞ Bernoulli utility function u with risk aversion bounds A^U and A^L that satisfy $\rho^L = A^L$ and $A^U \leq \rho^U + \varepsilon$.*

(c) If **OP.II** *has an optimal solution ρ^L, ρ^U with $\rho^L > 0$, then for all $0 < \varepsilon < \rho^L$, $\mathcal{D}$ is rationalizable with a strictly increasing, concave, C^∞ Bernoulli utility function u with risk aversion bounds A^U and A^L that satisfy $\rho^L - \varepsilon \leq A^L$ and $A^U = \rho^U$.*

Corollary 2. *A dataset $\mathcal{D} = (p_{t,s}, \pi_{t,s}, x_{t,s})_{t \leq T, s \leq S}$ is rationalizable by a CARA utility function if and only if the program* ***OP.II*** *has optimal value equal to 0.*

4.2 Subjective expected utility maximization

We now show how our methods can be generalized to encompass the subjective expected utility maximization model. We assume that state probabilities $\pi_{t,s}$ are constant across observations, i.e., $\pi_{t,s} = \pi_{v,s}$ for all observations $t, v \leq T$, but are unknown to the consumer and empirical analyst. We assume that the decision maker chooses the amounts $(x_{t,s})_{t \leq T, s \leq S}$ by maximising her subjective expected utility.

In particular, let ω_s be the subjective belief of the decision maker that state s will occur. According to the subjective expected utility maximisation model, at observation t, she will decide on the quantities $(x_{t,s})_{t \leq T, s \leq S}$ by maximising the following problem:

$$(x_{t,s})_{t \leq T, s \leq S} \in \arg\max_{(x_s)_{s \leq S}} \sum_{s \leq S} \omega_s u(x_s) \text{ s.t. } \sum_s p_{t,s} x_s \leq \sum_s p_{t,s} x_{t,s}. \tag{9}$$

Definition 5. *We say that $\mathcal{D} = (p_{t,s}, x_{t,s})_{t \leq T, s \leq S}$ is rationalizable by the Subjective Expected Utility (SEU) model with strictly increasing, concave, and C^1 Bernoulli utility function $u : \mathbb{R}_{++} \to \mathbb{R}$ if there exists subjective beliefs $(\omega_s)_{s \leq S}$ such that for all observations $t \leq T$ condition (9) is satisfied.*

In order to obtain testable implications, we replace the known value of $\ln(\pi_s)$ in the revealed preference conditions for EU rationalizability by the unknown variables $\kappa_s = \ln(\omega_s)$. For example, condition (5) then translates to:

$$x_{t,s} \leq x_{v,k} \text{ implies } \delta_t + \ln(p_{t,v}) - \kappa_s - \delta_v - \ln(p_{v,k}) + \kappa_k \leq R^u \left(\ln(x_{v,k}) - \ln(x_{t,s}\right). \quad (10)$$

and condition (6) becomes:

$$x_{t,s} \leq x_{v,k} \text{ implies } \delta_t + \ln(p_{t,s}) - \kappa_s - \delta_v - \ln(p_{v,k}) + \kappa_k \geq R^L \left(\ln(x_{v,k}) - \ln(x_{t,v})\right). \quad (11)$$

These are obviously necessary conditions. On the other hand, if we can find such values κ_s for all states $s \in S$ then we can always define:

$$\omega_s = \frac{\exp(\kappa_s)}{\sum_k \exp(\kappa_k)} > 0.$$

These values sum to one. By substituting them into our equations, we obtain effectively (5) and (6) with $\pi_{t,s}$ being replaced by ω_s. Our problem to find the minimal bound on the relative risk aversion is then modified in the following way:

OP.III: $\min_{\rho^U, \rho^L \geq 0, (\delta_t)_{t \in T}} \rho^U - \rho^L$ subject to Eqs. (10) and (11),

where we replaced R^U and R^L by ρ^U and ρ^L.

We can then obtain equivalent results as in Theorem 3, but for the SEU model.

Corollary 3. *Consider a data set $\mathcal{D} = (p_{t,s}, x_{t,s})_{t \leq T, s \leq S}$.*

(a) If $\mathcal{D}$ is SEU rationalizable with a strictly increasing, concave and C^2 Bernoulli utility function, $u : \mathbb{R}_{++} \to \mathbb{R}$ with risk aversion bounds R^U and R^L, then **OP.III** *has a feasible solution and its optimal values, ρ^U, ρ^L, satisfy $\rho^U - \rho^L \leq R^U - R^L$.*

(b) If **OP.III** *has an optimal solution ρ^L, ρ^U then for all $\varepsilon > 0$, $\mathcal{D}$ is SEU rationalizable with a strictly increasing, concave, C^∞ Bernoulli utility function u with risk aversion bounds R^U and R^L that satisfy $\rho^L = R^L$ and $R^U \leq \rho^U + \varepsilon$.*

(c) If **OP.III** *has an optimal solution ρ^L, ρ^U with $\rho^L > 0$, then for all $0 < \varepsilon < \rho^L$, $\mathcal{D}$ is SEU rationalizable with a strictly increasing, concave, C^∞ Bernoulli utility function u with risk aversion bounds R^U and R^L that satisfy $\rho^L - \varepsilon \leq R^L$ and $R^U = \rho^U$.*

A similar result holds for the measure of absolute risk aversion. Note that the conditions (10) and (11) remain linear in the unknowns, so **OP.III** can easily be implemented.

4.3 Restricting the income range

In some cases, one might be interested in bounding the level of risk aversion over a certain range of income levels. All procedures presented so far can easily be modified to accomplish this task. Consider an income range $[a, b] \subset \mathbb{R}_{++}$. We can then define:

$$R^U_{[a,b]} = \sup_{x \in [a,b]} R(x).$$

and

$$R^L_{[a,b]} = \inf_{x \in [a,b]} R(x).$$

Let us now determine the tightest bound for the gap $R^U_{[a,b]} - R^L_{[a,b]}$. We first introduce two additional variables μ_a and μ_b, which will representing the log marginal utilities at income levels a and b, respectively. Given this, condition (4) has to be augmented with the following requirements:

$$x_{t,s} \leq a \text{ implies } \mu_a \leq \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) \tag{12}$$

$$a \leq x_{t,s} \text{ implies } \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) \leq \mu_a. \tag{13}$$

and similarly for b:

$$x_{t,s} \leq b \text{ implies } \mu_b \leq \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right), \tag{14}$$

$$b \leq x_{t,s} \text{ implies } \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) \leq \mu_b. \tag{15}$$

Next, we modify constraints (5) and (6) by adding these two income levels a and b with their marginal utilities μ_a and μ_b. For example, to modify (5), we need to add the following conditions:

$$a \leq x_{t,s} \leq x_{v,k} \leq b \text{ implies } \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) - \delta_v - \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right) \leq R^U_{[a,b]}\left(\ln(x_{v,k}) - \ln(x_{t,s})\right). \tag{16}$$

together with:

$$a \leq x_{t,s} \leq b \text{ implies } \mu_a - \delta_t - \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) \leq R^U_{[a,b]}\left(\ln(x_{t,s}) - \ln(a)\right), \tag{17}$$

and

$$a \leq x_{t,s} \leq b \text{ implies } \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) - \mu_b \leq R^U_{[a,b]}\left(\ln(x_{t,s}) - \ln(b)\right). \tag{18}$$

An analogous adaptation for (6) applies to modify the constraints for $R^L_{[a,b]}$.

4.4 Including errors

In practice, the dataset $\mathcal{D}$ may fail to satisfy the (stringent) condition (4) required for expected-utility rationalizability. In that case, the feasibility problem **OP.I** has no solution and the sharp bounds it delivers are not directly available. Even when $\mathcal{D}$ does satisfy (4), however, it may still be undesirable to treat the observed choices as error-free: the decision maker may make small mistakes, and the researcher may observe choices with noise. This motivates an implementation that accommodates deviations from exact rationalizability and allows the bounds to be computed in the presence of errors.

Accordingly, this section develops a simple error-robust procedure for computing our bounds when exact feasibility is either violated or not compelling as a maintained assumption. Since the empirical application in the next section uses experimental data, we present the approach in an experimental-data setting; nonetheless, the logic is general and applies equally to other environments in which choices are measured imperfectly. Our treatment of errors is in the spirit of Echenique, Imai, and Saito (2023), who characterizes the minimal perturbation to perceived probabilities needed to rationalize observed choices by expected utility. Here, the role of the perturbation is to reconcile the model's exact restrictions with the inevitably noisy measurement of choice behavior.

The relevance of such errors is particularly transparent in laboratory data. In experiments, discrepancies between intended and recorded consumption choices can arise from limited attention, imperfect understanding of the task, and unsystematic implementation mistakes, all of which weaken the mapping from the elicited choice to the preference-driven choice the experiment seeks to measure. In our application, we use experimental data from Choi, Kariv, Müller, and Silverman (2014), where subjects choose from graphical depictions of two-dimensional budget lines. This interface can make it difficult to accurately perceive and trade off the underlying prices, so that part of the resulting deviation from exact rationalizability can be interpreted as measurement error induced by the experimental design—most naturally, as noise stemming from limited attention to prices.

In particular, we introduce multiplicative errors $\theta_{t,s}$ to the prices, $p_{t,s}$ in order to obtain the following modified decision problem:

$$\max_{(x_s)_{s\in S}} \sum_s \pi_{t,s} u(x_s) \quad \text{subject to} \quad \sum_s \theta_{t,s} p_{t,s} x_s \leq \sum_s \theta_{t,s} p_{t,s} x_{t,s}. \tag{19}$$

If $\theta_{t,s} = 1$, then there is no error and (19) reduces to the standard expected utility model. However, if $\theta_{t,s}$ differs from 1, it might be that the data is no longer expected utility rationalizable. The errors $\theta_{t,s} \neq 1$ may capture mistakes on behalf of the decision maker due to limited attention in prices. An alternative explanation could be that there is no error in prices but individuals have perception errors in the probabilities. This would give the following modified decision problem:

$$\max_{(x_s)_{s\in S}} \sum_s \frac{\pi_{t,s}}{\theta_{t,s}} u(x_s) \quad \text{subject to} \quad \sum_s p_{t,s} x_s \leq \sum_s p_{t,s} x_{t,s}. \tag{20}$$

where now $\frac{1}{\theta_{t,s}} \neq 1$ measures how far perceved probabilities deviate from the true probabilities. One can easily check that the first order conditions for (19) and (20) are identical, and will therefore lead to the same revealed preference conditions (see also Echenique, Imai, and Saito (2023) for further discussion).

In particular, the first order condition corresponding to the problem (19) or (20) are given by:

$$u'(x_s) = \lambda^t \frac{\theta_{t,s} p_{t,s}}{\pi_{t,s}}.$$

Condition (4) will then be modified to:

$$x_{t,s} \leq x_{v,k} \text{ implies } \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) - \delta_v - \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right) \geq \varepsilon_{v,k} - \varepsilon_{t,s}, \tag{21}$$

where we define $\varepsilon_{t,s} = \ln(\theta_{t,s})$. Whereas conditions (5) and (6) are modified to:

$$\begin{aligned} x_{t,s} \leq x_{v,k} \text{ implies } \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) - \delta_v - \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right) \\ \leq R^U \left(\ln(x_{v,k}) - \ln(x_{t,s})\right) + \varepsilon_{v,k} - \varepsilon_{t,s}. \end{aligned} \tag{22}$$

and

$$\begin{aligned} x_{t,s} \leq x_{v,k} \text{ then } \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) - \delta_v - \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right) \\ \geq R^L \left(\ln(x_{v,k}) - \ln(x_{t,s})\right) + \varepsilon_{v,k} - \varepsilon_{t,s}. \end{aligned} \tag{23}$$

Since $\varepsilon_{t,s} = 0$ if and only if $\theta_{t,s} = 1$, a natural way to assess (22) and (23) would be set the variables $(\varepsilon_{t,s})_{t\leq T, s\leq S}$ as close to zero as possible. An obvious candidate would be to minimize the sum of squares, $\sum_{t\leq T, s\leq S}(\varepsilon_{t,s})^2$. At the same time, however, we also want to minimize our usual objective function, being the distance between ρ^U and ρ^L. Taking both objectives under consideration, we opt for solving the following problem:

$$\textbf{OP.IV} \quad \min_{\rho^L,\rho^U\geq 0, (\varepsilon_{t,s},\delta_t)_{t\leq T,s\leq S}} \phi\left(\rho^U - \rho^L\right) + (1-\phi)\sum_{t,s}(\varepsilon_{t,s})^2 \text{ subject to (22) and (23).}$$

where we replace R^U and R^L by ρ^U and ρ^L.

Note that **OP.IV** amounts to minimizing a (convex) quadratic objective function subject to a set of linear constraints, and is therefore solvable in polynomial time.

The tuning parameter, ϕ, captures the trade off between minimizing the errors versus the difference between the risk aversion bounds. For $\phi \approx 1$, priority is given to closing the gap between ρ^U and ρ^L. This means that we are effectively looking for a CRRA utility function which is closest to rationalising the dataset $\mathcal{D}$ in terms of minimizing the squared errors $\sum_{s\leq S, t\leq T}(\varepsilon_{t,s})^2$. On the other hand, if we take $\phi \approx 0$, our first aim is to minimize the errors in order for the data set to be EU rationalizable. Only when this is done, we will try to minimize the gap between the risk aversion bounds. In the next section, we will provide results for varying levels of ϕ.

Finally, let us remark that given the results in section 4.1, it is straightforward to modify **OP.IV** to the absolute risk aversion setting by a simple modification of the terms $(\ln(x_{v,k}) - \ln(x_{t,s}))$ to $(x_{v,k} - x_{t,s})$ in equations (22) and (23).

5 Application

We illustrate our results using the large-scaled experimental data sets collected by Choi, Kariv, Müller, and Silverman (2014). In this experiment, subjects solved 25 questions. Each question showed a graphical illustration of a two-dimensional budget set over two state contingent goods, each occurring with probability 1/2. These probabilities were known to the subject prior to making choices. In each round, every subject selected an allocation of these state-contingent goods, by clicking on the budget line (i.e., forcing the entire income to be exhausted). The 25 budgets were randomly chosen across subjects (and questions).

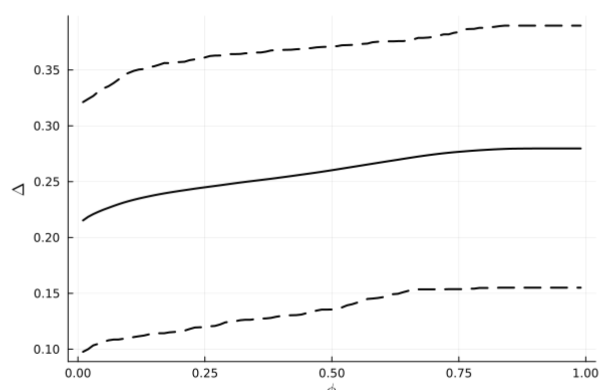
The sample contains data on subjects from the Dutch CentERpanel, which comprise over 2,000 households and 5,000 individuals. This forms a representative cross-section of the Dutch-speaking population in the Netherlands. After cleaning, we retain 1,182 subjects.

For every subject, we solve **OP.IV** for $\phi \in \{0.01, \ldots, 0.99\}$ which gives, for each choice of ϕ, optimal values of, ρ^U, ρ^L and $\theta_{t,s}$ for all 1,182 individuals in the sample. Recall that values of $\theta_{t,s}$ closer to 1, correspond to choices that are closer to being EU rationalizable. As such, we can assess the degree to which a subject violates EU maximization by assessing how far $\theta_{t,s}$ deviates from 1. This motivates the following goodness-of-fit measure:

$$\Delta = \frac{1}{TS} \sum_{t=1}^{T} \sum_{s=1}^{S} |1 - \theta_{t,s}|.$$

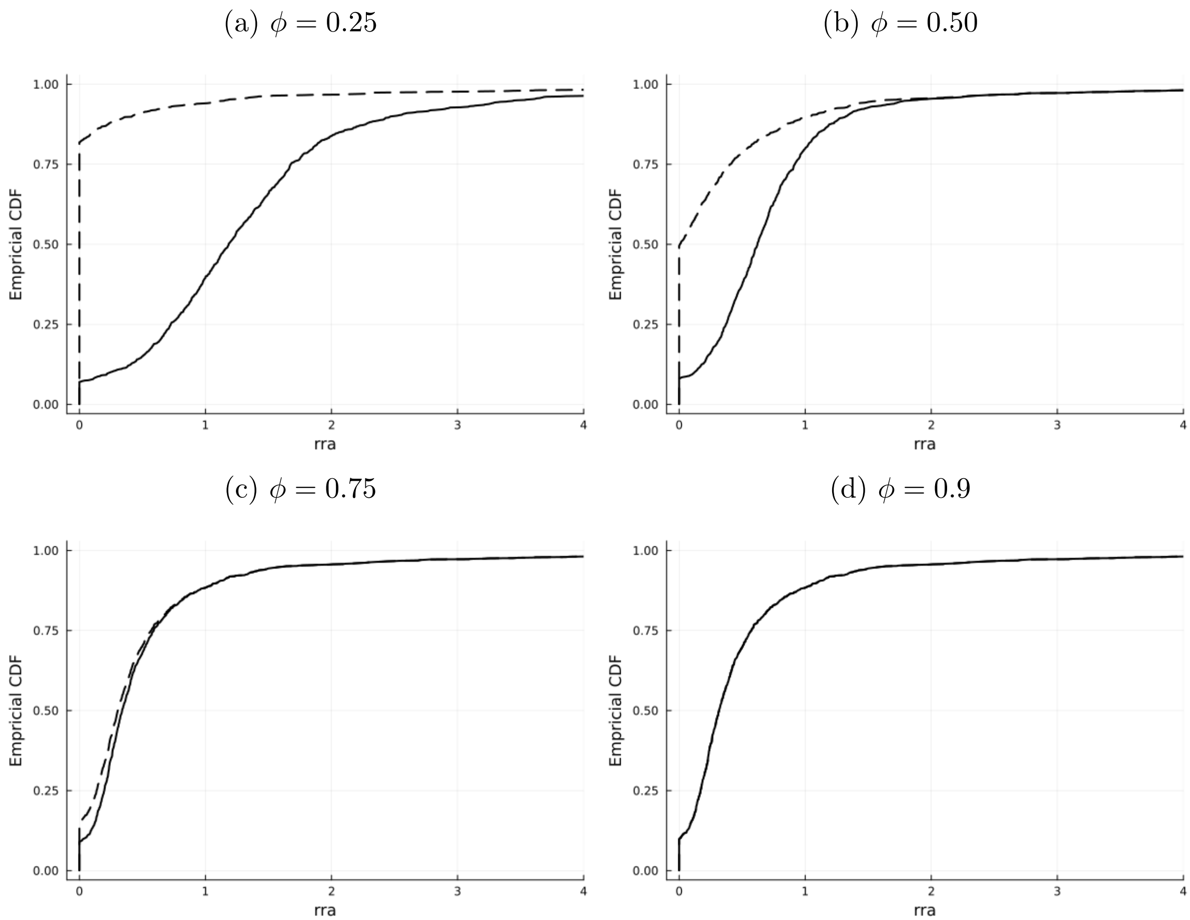
Figure 2 shows the mean, the 10th and 90th percentile of Δ over the subjets in the sample as a function of ϕ. As ϕ increases, more weight is given to minimizing the gap $\rho^U - \rho^L$ leading to a higher value for Δ. The average value of Δ is around 21% for $\phi = 0.01$ and increases to 28% for $\phi = 0.99$. Although we need sizeable values of Δ to EU rationalize the data, the increase in Δ when ϕ increases remains moderate. Hence, the amount of error needed to rationalize the data seem largely independent of the value of ϕ.

Figure 2: The average (solid line) and the 10th and 90th percentile (dashed lines) of Δ as function of ϕ



Next, Table 1 contains summary statistics of the values of ρ^L and ρ^U that solve **OP.IV** for $\phi = 0.25, 0.5, 0.75$ and 0.9. Figure 1 plots the empirical cumulative distributions of

Figure 3: Empirical CDF of ρ^L (dashed) and ρ^U (solid) for different values of ϕ



ρ^U and ρ^L over the sample of subjects.

As is clear from the figure and the table, the gap $\rho^U - \rho^L$ becomes smaller the higher ϕ (since increasing ϕ gives higher weight on minimizing the gap). However, even for low values of $\phi = 0.25$, the bounds are still informative with an average gap of 1.17. For this value of ϕ, we are unable to rule out that the lower bound on the relative risk aversion equals zero for a sizeable part of the population, as $\rho^L = 0$ for more than 75% of the subjects. The upper bound, however, contains more information. For example, for half of the subjects, ρ^U is no more than 1.08, while the mean upper bound is around 1.45. Higher values of ϕ give more informative bounds (at the cost of allowing for larger errors); for $\phi = 0.9$, the median level of relative risk aversion is about 0.3, while the average is 0.61, which is twice as large.

Our findings point to some interesting facts. First of all, our estimates of the relative

Table 1: Summary statistics of computed values for ρ^U and ρ^L for various values of ϕ.

		min	1st quartile	median	3rd quartile	max	mean
$\phi = 0.25$	ρ^L	0	0	0	0	20.076	0.279
	ρ^U	0	0.725	1.184	1.677	20.076	1.448
	$\rho^U - \rho^L$	0	0.577	1.079	1.603	7.511	1.170
$\phi = 0.5$	ρ^L	0	0	0.006	0.4044	20.076	0.460
	ρ^U	0	0.371	0.621	0.918	20.076	0.841
	$\rho^U - \rho^L$	0	0	0.345	0.642	2.262	0.381
$\phi = 0.75$	ρ^L	0	0.136	0.305	0.569	20.076	0.599
	ρ^U	0	0.198	0.339	0.589	20.076	0.630
	$\rho^U - \rho^L$	0	0	0	0	0.714	0.0318
$\phi = 0.9$	ρ^L	0	0.170	0.316	0.573	20.075	0.613
	ρ^U	0	0.170	0.316	0.573	20.075	0.613
	$\rho^U - \rho^L$	0	0	0	0	0.0145	0.00002

risk aversion are quite low in comparison to the standard values used for calibration, which usually takes values of 2 or higher. On the other hand, they do not seem to be that implausible given the ballpark of estimates in the literature obtained from Euler equations estimations (see Elminejad, Havranek, and Irsova (forthcoming) for a recent survey). Our estimates also come quite close to Chetty (2006) who estimated a mean risk aversion of 0.71 from data on labor supply. In his paper, he mentions that one of the interpretations of his finding is that:

> ...it provides new evidence against canonical expected utility theory as a descriptive model of choice uncertainty.

Our results, on the other hand, show that even when assuming EU, but using nonparametric revealed preference methods, it is possible to obtain upper bounds on relative risk aversion that are quite low.

Finally, we want to indicate that there is quite some heterogeneity in our estimates of the relative risk aversion. For a sizeable part of the population the relative risk aversion seems to be quite close to zero, suggesting risk-neutral behavior. For example, when choosing $\phi = 0.9$, about 10% of the subjects have bounds ρ^U and ρ^L that are very close to zero. The distribution of risk aversion also seem to be skewed to the right, with a mean that is twice the median (for higher values of ϕ). So, although for many subjects, relative risk aversion is quite low, there are some subject outliers with quite high levels of risk aversion.

6 Conclusions

We presented a non-parametric revealed-preference framework to obtain the tightest (uniform) bounds on coefficients of risk aversion in the expected utility model with Arrow–Debreu securities, allowing state probabilities to be either known (EU) or unknown (SEU). The resulting bounds can be computed using off-the-shelf linear or quadratic optimization algorithms, which makes the approach practical even for large datasets. In our application to a large experimental dataset, we find that bounds on relative risk aversion are often quite tight, while risk aversion remains heterogeneous across subjects;

mean levels are generally below 2 and mostly below 1.

Two extensions appear particularly promising. First, our analysis focuses on Arrow–Pratt curvature, but the same revealed-preference logic could be pushed to higher-order risk attitudes such as prudence and temperance (involving u''' and u''''). Conceptually, these objects impose shape restrictions on how (log) marginal utility changes with income beyond the second derivative. Methodologically, this suggests augmenting our programs with additional inequality constraints that bound higher-order derivatives (or suitable finite-difference analogues) over the observed consumption support, yielding sharp bounds on higher-order attitudes under minimal smoothness assumptions.

Second, while we work in an Arrow–Debreu environment, this is not as restrictive as it may seem: when asset payoffs span states, choices over general assets can be transformed into an observationally equivalent Arrow–Debreu representation. In many economic applications, however, markets are incomplete and observed assets do not span the state space. Extending the framework to such settings would require treating state prices (or equivalent stochastic discount factors) as partially identified objects consistent with no-arbitrage and observed payoffs, and then integrating this additional layer of constraints into our LP/QP programs. Doing so would broaden the applicability of the method from experimental Arrow–Debreu choices to portfolio data with general payoffs in incomplete markets.

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A Proof of Theorem 2

The first part of the proof follows from the derivations in the main text. Next, let $(\delta_t)_{t\leq T}$ be a set of values such that (4) is satisfied. Consider a plot of the values $(x_{t,s}, y_{t,s})$ where $y_{t,s} = e^{\delta_t}\left(\frac{p_{t,s}}{\pi_{t,s}}\right)$. Consider the function f that linearly interpolates these points. If there are indices t and s such that $x_{t,s} = 0$, define $f(0) = \min\{y_{t,s}\}$ where the minimum is taken over all indices t and s satisfying $x_{t,s} = 0$.

If $x_{t,s} > 0$ for all t and s, we extend f in a continuous way to the left by a constant function equal to the largest value of $y_{t,s}$. We also extend f in a continuous way to the right by a constant function equal to the smallest value of $y_{t,s}$. The function f obtained in this way is strictly positive, continuous and non-increasing.

Fix an arbitrary value $\hat{x} > 0$ and consider the function $u(x) = \int_{\hat{x}}^{x} f(z)dz$. This function is concave (as $u'(x) = f(x)$ is non-increasing), C^1, and strictly increasing. Let us show that it provides an expected utility rationalization for the dataset. Let $m_t = \sum_{s\leq S} p_{t,s}x_{t,s}$ be the total expenditure at observation t, and consider a bundle $(x_s)_{s\leq S}$ such that $\sum_{s\leq S} p_{t,s}x_s \leq m_t$. Then, by concavity:

$$
\begin{aligned}
\sum_{s\leq S}\pi_{t,s}(u(x_s) - u(x_{t,s})) &\leq \sum_{s\leq S}\pi_{t,s}u'(x_{t,s})(x_s - x_{t,s}),\\
&= \sum_{s\in S}\pi_{t,s}f(x_{t,s})(x_s - x_{t,s}),\\
&\leq \sum_{s\in S}\pi_{t,s}\left[e^{\delta_t}\frac{p_{t,s}}{\pi_{t,s}}\right](x_s - x_{t,s}),\\
&= e^{\delta_t}\sum_{s\in S}p_{t,s}(x_s - x_{t,s}) \leq 0.
\end{aligned}
$$

As such, $\mathcal{D}$ is EU rationalizable by the utility function u.

B Proof of Theorem 3

(a). From the text, we have argued that conditions (5)-(6) are feasible for the linear programming problem **OP.I** By optimality, we must then have that $\rho^U - \rho^L \leq R^U - R^L$.

(b). Let ρ^U, ρ^L and $(\delta_t)_{t\leq T}$ be the optimal values for **OP.I**. Consider the graph where we plot values of $y_{t,s} \equiv \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right)$ against the values of $z_{t,s} \equiv \ln(x_{t,s})$. Given that for

$x_{t,s} \leq x_{v,s}$

$$\delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) - \delta_v - \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right) \leq \rho^U(\ln(x_{v,k}) - \ln(x_{t,s})),$$
$$\delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right) - \delta_v - \ln\left(\frac{p_{v,k}}{\pi_{v,k}}\right) \geq \rho^L(\ln(x_{v,k}) - \ln(x_{t,s})).$$

holds for all $x_{t,s} \leq x_{v,k}$, the absolute value of the piecewise linear function connecting these points has slope whose absolute value is in the interval $[\rho^L, \rho^U]$. (note that for $x_{t,s} = x_{v,k}$ also $y_{t,s} = y_{v,k}$).

Let us make this a bit more formal. For notational convenience, enlist the numbers $z_{t,s} = \ln(x_{t,s})$ from smallest to largest, $z_{(0)} < \ldots < z_{(n)}$. Let $y_{(i)} = \delta^*_{(i)} + \ln\left(\frac{p_{(i)}}{\pi_{(i)}}\right)$ where $\delta_{(i)}, p_{(i)}$ and $\pi_{(i)}$ are the corresponding values of $\delta_{t,s}, p_{t,s}$ and $\pi_{t,s}$ for $x_{t,s} = x_{(i)}$.

For $i \geq 1$. Let $s_{(i)} = \dfrac{y_{(i-1)} - y_{(i)}}{z_{(i)} - z_{(i-1)}}$ be the absolute value of the slope of the segment connecting the point $(z_{(i-1)}, y_{(i-1)})$ with $(z_{(i)}, y_{(i)})$. Pick any $s_{(0)} \in]\rho^L, \rho^U[$ and any $s_{(n+1)} \in]\rho^L, \rho^U[$.

Now let f be the linear interpolation that connects the points

$$(z_{(0)}, y_{(0)}), (z_{(1)}, y_{(1)}), \ldots, (z_{(n)}, y_{(n)}).$$

We extend f to the left by connecting a line segment to the left of $(z_{(0)}, y_{(0)})$ with slope $s_{(0)}$ and we extend f to the right by connecting a line segment to the right of $(z_{(n)}, y_{(n)})$ with slope $s_{(n+1)}$.

The function f has the following properties:

- It is non-increasing
- It contains the points $(z_{(i)}, y_{(i)})$
- It segments have slopes with absolute value in the interval $[\rho^L, \rho^U]$.

The function f, however, is not C^1. In order to fix this, we will use convolution techniques. Before doing this, however, it is necessary to transform f into a new function that still satisfies the first and second property above, but does not have any kinks at the points $(z_{(i)}, y_{(i)})$. To do this, we need to slightly relax the third property towards the condition that the function has no slopes with absolute values in the interval $[\rho^L, \rho^U + \varepsilon]$.

We fix a $\varepsilon > 0$ with

$$\varepsilon < \frac{1}{2}\min_i\{s_{(i)}|s_{(i)} > \rho^L\} \text{ and } \varepsilon < \min_i\{s_{(i)} - s_{(i-1)}|s_{(i)} - s_{(i-1)} > 0\}.$$

The transformation of the function f takes 3 steps. The first transforms f into a function $\widehat{f}$. The second step transforms $\widehat{f}$ into $\overline{f}$ and the third uses convolution to transform $\overline{f}$ into f^*.

Step 1:

The first modification relates to the endpoints of the segments of the interpolation f with slope $s_{(i)} = \rho^L$. For these, we are going to move the points $z_{(i)}$ slightly either to the right or to the left. Let $\delta > 0$ be small enough such that for all $i = 1, \ldots, n$ with $s_{(i)} > \rho^L$:

$$\begin{aligned}
&0 < \delta \frac{s_{(i)} - \rho^L}{z_{(i)} - z_{(i-1)} - \delta} < \frac{\varepsilon}{2}, \\
&0 < 2\delta \frac{s_{(i)} - \rho^L}{z_{(i)} - z_{(i-1)} - 2\delta} < \frac{\varepsilon}{2}. \\
&\delta < \min_i \frac{z_{(i)} - z_{(i-1)}}{3}.
\end{aligned}$$

Consider the following procedure to set the points $(\widehat{z}_{(i)}, \widehat{y}_{(i)})$, $i = 0, \ldots, n$.

1. If $s_{(i)} = \rho^L$ and $s_{(i+1)} > \rho^L$ define $\widehat{z}_{(i)} = z_{(i)} + \delta$ and $\widehat{y}_{(i)} = y_{(i)} - \delta\rho^L$
2. If $s_{(i)} > \rho^L$ and $s_{(i+1)} = \rho^L$ define $\widehat{z}_{(i)} = z_{(i)} - \delta$ and $\widehat{y}_{(i)} = y_{(i)} + \delta\rho^L$
3. In all other cases, define $\widehat{z}_{(i)} = z_{(i)}$ and $\widehat{y}_{(i)} = y_{(i)}$.

We define the slopes $\widehat{s}_{(i)} = \dfrac{\widehat{y}_{(i-1)} - \widehat{y}_{(i)}}{\widehat{z}_{(i)} - \widehat{z}_{(i-1)}}$ for $i = 1, \ldots, n$ and we set $\widehat{s}_{(0)} = s_{(0)}$ and $\widehat{s}_{(n+1)} = s_{(n+1)}$.

Let $\widehat{f}$ be the function that linearly interpolates the points

$$(\widehat{z}_{(0)}, \widehat{y}_{(0)}), (\widehat{z}_{(1)}, \widehat{y}_{(1)}), \ldots, (\widehat{z}_{(n)}, \widehat{y}_{(n)}).$$

Extend $\widehat{f}$ to the left by adding a half-line with slope $-\widehat{s}_{(0)}$ to the point $(\widehat{z}_{(0)}, \widehat{y}_{(0)})$ and add the half-line with slope $-s_{(n+1)}$ to the right of point $(\widehat{z}_{(n)}, \widehat{y}_{(n)})$.

The function $\widehat{f}$ is very similar to f but it extends the segments with slopes ρ^L by a distance δ to the left and to the right.

The following shows some properties of $\widehat{f}$.

Lemma 1. *For all i,*

- $s_{(i)} + \frac{\varepsilon}{2} \geq \widehat{s}_{(i)} \geq s_{(i)}$
- $\widehat{s}_{(i)} > \rho^L$ *if and only if* $s_{(i)} > \rho^L$.
- *For all i,* $\widehat{f}(z_{(i)})) = y_{(i)}$.

Proof. Note that for $i = 0$ and $i = n + 1$, we have $\widehat{s}_{(0)} = s_{(0)}$ and $\widehat{s}_{(n+1)} = s_{(n+1)}$ (by definition), so the first two properties are satisfied for $i = 0, n+1$. To check the first two properties for all other $1 < i < n + 1$, we consider 8 cases.

1. If $s_{(i-1)} > \rho^L$, $s_{(i)} > \rho^L$ and $s_{(i+1)} > \rho^L$ then $\widehat{z}_{(i-1)} = z_{(i-1)}, \widehat{z}_{(i)} = z_{(i)}$, $\widehat{y}_{(i-1)} = y_{(i-1)}$ and $\widehat{y}_{(i)} = y_{(i)}$, so $\widehat{s}_{(i)} = s_{(i)}$. Evidently, the second claim also hold.

2. If $s_{(i-1)} > \rho^L$, $s_{(i)} > \rho^L$ and $s_{(i+1)} = \rho^L$ then $\widehat{z}_{(i-1)} = z_{(i-1)}, \widehat{z}_{(i)} = z_{(i)} - \delta$, $\widehat{y}_{(i-1)} = y_{(i-1)}$ and $\widehat{y}_{(i)} = y_{(i)} + \delta\rho^L$. As such,
$$\begin{aligned}\widehat{s}_{(i)} &= \frac{y_{(i-1)} - y_{(i)} - \delta\rho^L}{z_{(i)} - \delta - z_{(i-1)}},\\ &= \frac{s_{(i)}(z_{(i)} - z_{(i-1)} - \delta) + \delta(s_{(i)} - \rho^L)}{z_{(i)} - z_{(i-1)} - \delta},\\ &= s_{(i)} + \delta\frac{s_{(i)} - \rho^L}{z_{(i)} - z_{(i-1)} - \delta},\end{aligned}$$
The latter is strictly between $s_{(i)}$ and $s_{(i)} + \frac{\varepsilon}{2}$, so the first property holds. Note that $\widehat{s}_{(i)} > \rho^L$, so also the second property holds.

3. If $s_{(i-1)} > \rho^L$, $s_{(i)} = \rho^L$ and $s_{(i+1)} > \rho^L$, then: $\widehat{z}_{(i-1)} = z_{(i-1)} - \delta$, $\widehat{z}_{(i)} = z_{(i)} + \delta$, $\widehat{y}_{(i-1)} = y_{(i)} + s_{(i)}\delta, \widehat{y}_{(i)} = y_{(i)} - s_{(i)}\delta$. As such, we have that $\widehat{s}_{(i)} = s_{(i)} = \rho^L$, so the two properties are satisfied.

4. If $s_{(i-1)} > \rho^L$, $s_{(i)} = \rho^L$ and $s_{(i+1)} = \rho^L$, then: $\widehat{z}_{(i-1)} = z_{(i-1)} - \delta$, $\widehat{z}_{(i)} = z_{(i)}$, $\widehat{y}_{(i-1)} = y_{(i)} + s_{(i)}\delta$, $\widehat{y}_{(i)} = y_{(i)}$. As such, we have that
$$\begin{aligned}\widehat{s}_{(i)} &= \frac{y_{(i-1)} + \delta\rho^L - y_{(i)}}{z_{(i)} - z_{(i-1)} + \delta},\\ &= \frac{s_{(i)}(z_{(i)} - z_{(i-1)} + \delta)}{z_{(i)} - z_{(i-1)} + \delta} = s_{(i)}.\end{aligned}$$
Conclude that $\widehat{s}_{(i)} = s_{(i)} = \rho^L$.

5. If $s_{(i-1)} = \rho^L$, $s_{(i)} > \rho^L$ and $s_{(i+1)} > \rho^L$, then: $\widehat{z}_{(i-1)} = z_{(i-1)} + \delta$, $\widehat{z}_{(i)} = z_{(i)}$, $\widehat{y}_{(i-1)} = y_{(i-1)} - \delta\rho^L$ and $\widehat{y}_{(i)} = y_{(i)}$, so:

$$\begin{aligned}\widehat{s}_{(i)} &= \frac{y_{(i-1)} - \delta\rho^L - y_{(i)}}{z_{(i)} - z_{(i-1)} - \delta}, \\ &= \frac{s_{(i)}[z_{(i)} - z_{(i-1)} - \delta] + \delta(s_{(i)} - \rho^L)}{z_{(i)} - z_{(i-1)} - \delta}, \\ &= s_{(i)} + \delta\frac{s_{(i)} - \rho^L}{z_{(i)} - z_{(i-1)} - \delta}.\end{aligned}$$

This is strictly between $s_{(i)}$ and $s_{(i)} + \frac{\varepsilon}{2}$. As such, $\widehat{s}_{(i)} > \rho^L$.

6. If $s_{(i-1)} = \rho^L$, $s_{(i)} > \rho^L$ and $s_{(i+1)} = \rho^L$. Then: $\widehat{z}_{(i-1)} = z_{(i-1)} + \delta$, $\widehat{z}_{(i)} = z_{(i)} - \delta$, $\widehat{y}_{(i-1)} = y_{(i-1)} - \delta\rho^L$ and $\widehat{y}_{(i)} = y_{(i)} + \delta\rho^L$. Then:

$$\begin{aligned}\widehat{s}_{(i)} &= \frac{y_{(i-1)} - \delta\rho^L - y_{(i)} - \delta\rho^L}{z_{(i)} - z_{(i-1)} - 2\delta}, \\ &= \frac{s_{(i)}(z_{(i)} - z_{(i-1)} - 2\delta) + 2\delta(s_{(i)} - \rho^L)}{z_{(i)} - z_{(i-1)} - 2\delta} \\ &= s_{(i)} + 2\delta\frac{s_{(i)} - \rho^L}{z_{(i)} - z_{(i-1)} - 2\delta}.\end{aligned}$$

The latter is strictly between $s_{(i)}$ and $s_{(i)} + \frac{\varepsilon}{2}$. Conclude that $\widehat{s}_{(i)} > \rho^L$

7. If $s_{(i-1)} = \rho^L$, $s_{(i)} = \rho^L$ and $s_{(i+1)} > \rho^L$. Then: $\widehat{z}_{(i-1)} = z_{(i-1)}$, $\widehat{z}_{(i)} = z_{(i)} + \delta$, $\widehat{y}_{(i-1)} = y_{(i-1)}$ and $\widehat{y}_{(i)} = y_{(i)} - \delta\rho^L$. So:

$$\widehat{s}_{(i)} = \frac{y_{(i-1)} - y_{(i)} + \delta\rho^L}{z_{(i)} - z_{(i-1)} + \delta}, \tag{24}$$

$$= \frac{s_{(i)}[z_{(i)} - z_{(i-1)} + \delta]}{z_{(i)} - z_{(i-1)} + \delta} = s_{(i)}. \tag{25}$$

Then $\widehat{s}_{(i)} = s_{(i)} = \rho^L$, so the two properties are satisfied.

8. If $s_{(i-1)} = \rho^L$, $s_{(i)} = \rho^L$ and $s_{(i+1)} = \rho^L$ then: $\widehat{z}_{(i-1)} = z_{(i-1)}, \widehat{z}_{(i)} = z_{(i)}, \widehat{y}_{(i-1)} = y_{(i-1)}$ and $\widehat{y}_{(i)} = \widehat{y}_{(i)}$, so $\widehat{s}_{(i)} = s_{(i)} = \rho^L$. The two properties are satisfied.

Let us now show that for all i, $\widehat{f}(z_{(i)}) = y_{(i)}$. If $z_{(i)} = \widehat{z}_{(i)}$ then this is obvious as $\widehat{f}$ interpolates the points $(\widehat{z}_{(i)}, \widehat{y}_{(i)})$.

- If $\widehat{z}_{(i)} = z_{(i)} - \delta$ then we know that $s_{(i)} > \rho^L$ and $s_{(i+1)} = \widehat{s}_{(i)} = \rho^L$. The segment connecting $(\widehat{z}_{(i)}, \widehat{y}_{(i)})$ and $(\widehat{z}_{(i+1)}, \widehat{y}_{(i+1)})$ has slope with absolute value ρ^L. As $\widehat{z}_{(i)} < z_{(i)} < \widehat{z}_{(i+1)}$ and $y_{(i)} = \widehat{y}_{(i)} - \delta\rho^L$ we can conclude that $\widehat{f}(z_{(i)}) = y_{(i)}$.

- If $\widehat{z}_{(i)} = z_{(i)} + \delta$ then we know that $s_{(i)} = \widehat{s}_{(i)} = \rho^L$ and $s_{(i-1)} > \rho^L$. The segment of f between $(\widehat{z}_{(i-1)}, \widehat{y}_{(i-1)})$ and $(\widehat{z}_{(i)}, \widehat{y}_{(i)})$ has slope ρ^L. As $\widehat{z}_{(i-1)} < z_{(i)} < \widehat{z}_{(i+1)}$ and $y_{(i)} = \widehat{y}_{(i)} + \delta\rho^L$, we effectively have that $\widehat{f}(z_{(i)}) = y_{(i)}$.

$\square$

Step 2:

In this step, we are transforming the function $\widehat{f}$ into a new piecewise linear function $\overline{f}$ by marginally changing the slopes of the segments through the points $(\widehat{z}_{(i)}, \widehat{y}_{(i)})$.

Let $\overline{s}_{(0)} = \widehat{s}_{(0)} + \frac{\varepsilon}{2} \in [\rho^L, \rho^U + \varepsilon]$ and recursively set:

$$\overline{s}_{(i)} = \begin{cases} \widehat{s}_{(i)} - \frac{\varepsilon}{2} & \text{if } \overline{s}_{(i-1)} > \widehat{s}_{(i)} > \rho^L \\ \widehat{s}_{(i)} & \text{if } \widehat{s}_{(i)} = \rho^L \\ \widehat{s}_{(i)} + \frac{\varepsilon}{2} & \text{if } \overline{s}_{(i-1)} < \widehat{s}_{(i)} \\ \widehat{s}_{(i)} & \text{if } \overline{s}_{(i-1)} = \widehat{s}_{(i)}. \end{cases}$$

As $\widehat{s}_{(i)} \in [\rho^L, \rho^U + \frac{\varepsilon}{2}]$, and $\widehat{s}_{(i)} > \rho^L$ iff $s_{(i)} > \rho^L$, we see that $\overline{s}_{(i)} \in [\rho^L, \rho^U + \varepsilon]$

For all i, define the linear function:

$$\overline{f}_{(i)}(z) = \widehat{y}_{(i)} + \overline{s}_{(i)}(\widehat{z}_{(i)} - z),$$

This linear function passes through the point $(\widehat{z}_{(i)}, \widehat{y}_{(i)})$ and has slope $-\overline{s}_{(i)}$.

Lemma 2. *The following holds*

- $\overline{s}_{(i)} > \rho^L$ *if and only if* $s_{(i)} > \rho^L$.
- *For all* $i = 1, \ldots, n$, *the (linear) functions* $\overline{f}_{(i-1)}(z)$ *and* $\overline{f}_{(i)}(z)$ *intersect at (at least one) value of* z *between* $\widehat{z}_{(i-1)}$ *and* $\widehat{z}_{(i)}$.
- *The intersection (can be taken to be) strictly between* $\max\{\widehat{z}_{(i-1)}, z_{(i-1)}\}$ *and* $\min\{\widehat{z}_{(i)}, z_{(i)}\}$ *unless* $\overline{s}_{(i)} = \rho^L$ *and* $\overline{s}_{(i-1)} > \rho^L$. *In that case the intersection is at the point* $(\widehat{z}_{(i-1)}, \widehat{y}_{(i-1)})$.

Proof. The first is easily checked. For the other two, we go over 4 different cases.

We consider four cases.

1. $\overline{s}_{(i-1)} > \widehat{s}_{(i)} > \rho^L$.

   In that case, $\overline{s}_{(i)} = \widehat{s}_{(i)} - \frac{\varepsilon}{2}$. Let z be such that $\overline{f}_{(i-1)}(z) = \overline{f}_{(i)}(z)$. Then:

$$\begin{aligned}
&\widehat{y}_{(i-1)} + \overline{s}_{(i-1)}(\widehat{z}_{(i-1)} - z) = \widehat{y}_{(i)} + \widehat{s}_{(i)}(\widehat{z}_{(i)} - z) - \frac{\varepsilon}{2}(\widehat{z}_{(i)} - z),\\
\leftrightarrow &\widehat{s}_{(i)}(\widehat{z}_{(i)} - \widehat{z}_{(i-1)}) + \overline{s}_{(i-1)}(\widehat{z}_{(i-1)} - z) = \widehat{s}_{(i)}(\widehat{z}_{(i)} - z) - \frac{\varepsilon}{2}(\widehat{z}_{(i)} - z),\\
\leftrightarrow &z = \frac{\widehat{z}_{(i)}\varepsilon/2 + \widehat{z}_{(i-1)}[\overline{s}_{(i-1)} - \widehat{s}_{(i)}]}{\overline{s}_{(i-1)} - \widehat{s}_{(i)} + \varepsilon/2}.
\end{aligned}$$

   This shows that z is a weighted average of $\widehat{z}_{(i-1)}$ and $\widehat{z}_{(i)}$ with strict positive weights. As such, it is strictly between these two points.

   Also, as $s_{(i)} > \rho^L$, $\widehat{z}_{(i-1)} \geq z_{(i-1)}$ and $\widehat{z}_{(i)} \leq z_{(i)}$, the intersection is also strictly between $\max\{\widehat{z}_{(i-1)}, z_{(i-1)}\}$ and $\min\{\widehat{z}_{(i)}, z_{(i)}\}$.

2. If $\widehat{s}_{(i)} = \rho^L$, then $\overline{s}_{(i)} = \rho^L$. If $\overline{s}_{(i-1)} = \rho^L$ then $\overline{f}_{(i)}$ and $\overline{f}_{(i-1)}$ are identical functions passing through $(\widehat{z}_{(i)}, \widehat{y}_{(i)})$ and $(\widehat{z}_{(i-1)}, \widehat{y}_{(i-1)})$. As such, we can take any z strictly between these two values.

   Take now the case where $\overline{s}_{(i-1)} > \rho^L$.

   Let z be such that $\overline{f}_{(i-1)}(z) = \overline{f}_{(i)}(z)$. Then:

$$\begin{aligned}
&\widehat{y}_{(i-1)} + \overline{s}_{(i-1)}(\widehat{z}_{(i-1)} - z) = \widehat{y}_{(i)} + \rho^L(\widehat{z}_{(i)} - z),\\
\leftrightarrow &\rho^L(\widehat{z}_{(i)} - \widehat{z}_{(i-1)}) + \overline{s}_{(i-1)}(\widehat{z}_{(i-1)} - z) = \rho^L(\widehat{z}_{(i)} - z),\\
\leftrightarrow &z = \widehat{z}_{(i-1)}
\end{aligned}$$

   So we see that if $\overline{s}_{(i)} = \rho^L$ and $\overline{s}_{(i-1)} > \rho^L$ then the intersection is at the point $(\widehat{z}_{(i-1)}, \widehat{y}_{(i-1)})$.

3. If $\overline{s}_{(i-1)} < \widehat{s}_{(i)}$. $\overline{s}_{(i)} = \widehat{s}_{(i)} + \varepsilon/2$.

   Let z be such that $\overline{f}_{(i-1)}(z) = \overline{f}_{(i)}(z)$. Then:

$$\begin{aligned}
&\widehat{y}_{(i-1)} + \overline{s}_{(i-1)}(\widehat{z}_{(i-1)} - z) = \widehat{y}_{(i-1)} + \widehat{s}_{(i)}(\widehat{z}_{(i)} - z) + \frac{\varepsilon}{2}(\widehat{z}_{(i)} - z),\\
\leftrightarrow &\widehat{s}_{(i)}(\widehat{z}_{(i)} - \widehat{z}_{(i-1)}) + \overline{s}_{(i-1)}(\widehat{z}_{(i-1)} - z) = \widehat{s}_{(i)}(\widehat{z}_{(i)} - z) + \frac{\varepsilon}{2}(\widehat{z}_{(i)} - z),\\
\leftrightarrow &z = \frac{\widehat{z}_{(i-1)}[\widehat{s}_{(i)} - \overline{s}_{(i-1)}] + \widehat{z}_{(i)}\varepsilon/2}{\widehat{s}_{(i)} - \overline{s}_{(i-1)} + \varepsilon/2}.
\end{aligned}$$

As such, z is a weighted average of $\widehat{z}_{(i-1)}$ and $\widehat{z}_{(i)}$ with strict positive weights. This means that z is strictly between these two values. As $s_{(i)} > \rho^L$, we see that $\widehat{z}_{(i-1)} \geq z_{(i-1)}$ and $\widehat{z}_{(i)} \leq z_{(i)}$. This means that the intersection is also strictly between $\max\{\widehat{z}_{(i-1)}, z_{(i-1)}\}$ and $\min\{\widehat{z}_{(i)}, z_{(i)}\}$.

4. If $\widehat{s}_{(i)} = \overline{s}_{(i-1)}$, then $\overline{s}_{(i)} = \widehat{s}_{(i)}$. In this case the functions $\overline{f}_{(i)}$ and $\overline{f}_{(i-1)}$ are identical. Take z to be any value strictly between $\max\{\widehat{z}_{(i-1)}, z_{(i-1)}\}$ and $\min\{\widehat{z}_{(i)}, z_{(i)}\}$.

□

Given this lemma, we define by $w_{(i)}$ the value of z where $\overline{f}_{(i)}(z)$ and $\overline{f}_{(i-1)}(z)$. Previous lemma shows that $\widehat{z}_{(i-1)} \leq w_{(i)} \leq \widehat{z}_{(i)}$. Let $\kappa_{(i)} = \overline{f}_{(i)}(w_{(i)}) = \overline{f}_{(i-1)}(w_{(i)})$.

Define the function $\overline{f}$ to be the linear interpolations of the points

$$(w_{(1)}, \kappa_{(1)}), (w_{(2)}, \kappa_{(2)}), \ldots, (w_{(n-1)}, \kappa_{(n-1)}), (w_{(n)}, \kappa_{(n)}).$$

Extend $\overline{f}$ to the left by adjoining the line with slope $-\overline{s}_{(0)}$ to the point $(w_{(1)}, \kappa_{(0)})$ and to the right by adjoining the half-line with slope $-\overline{s}_{(n+1)}$ to the point $(w_{(n)}, \kappa_{(n)})$.

Note that the slopes of all segments of $\overline{f}$ are in the interval $[\rho^L, \rho^U + \varepsilon]$.

Lemma 3. *The following holds:*

- *For all i, $\overline{f}(z_{(i)}) = y_{(i)}$.*
- *The function has no corner points at $(z_{(i)}, y_{(i)})$, i.e. for all i and j, $z_{(i)} \neq w_{(j)}$*

Proof. If $z_{(i)} = \widehat{z}_{(i)}$ then $y_{(i)} = \widehat{y}_{(i)}$.

- If $i = 0$, then $f_{(0)}(w_{(1)}) = y_{(0)} + \overline{s}_{(1)}(\widehat{z}_{(0)} - w_{(1)}) = \kappa_{(1)}$. As $z_{(0)} \leq w_{(1)}$ we see that $(z_{(0)}, y_{(0)})$ is on the line segment to the left of $(w_{(1)}, \kappa_{(1)})$ with slope $\overline{s}_{(0)}$.

- if $i = n$, then $f_{(n)}(w_{(n)}) = y_{(n)} + \overline{s}_{(n)}(\widehat{z}_{(n)} - w_{(n)}) = \kappa_{(n)}$. As $w_{(n)} \leq z_{(n)}$, we see that $(z_{(n)}, y_{(n)})$ is on the line segment on the right of $(w_{(n)}, \kappa_{(n)})$ with slope $\overline{s}_{(n)}$.

- In any other case, $w_{(i)} \leq z_{(i)} \leq w_{(i+1)}$. Then $\overline{f}_{(i)}(w_{(i)}) = y_{(i)} + \overline{s}_{(i)}(z_{(i)} - w_{(i)}) = \kappa_{(i)}$ and $\overline{f}_{(i)}(w_{(i+1)}) = y_{(i)} + \overline{s}_{(i)}(z_{(i)} - w_{(i+1)}) = \kappa_{(i+1)}$. This means the line segment between $(w_{(i)}, \kappa_{(i)})$ and $(w_{(i+1)}, \kappa_{(i+1)}$ has slope $\overline{s}_{(i)}$. As $w_{(i)} \leq z_{(i)} \leq w_{(i+1)}$ and $y_{(i)} = \kappa_{(i)} + \overline{s}_{(i)}(w_{(i)} - z_{(i)})$ we see that $(z_{(i)}, y_{(i)})$ is also on this line segment.

Now take the case where $z_{(i)} \neq \widehat{z}_{(i)}$. There are two cases.

- if $s_{(i)} = \overline{s}_{(i)} = \rho^L$ and $s_{(i+1)} > \rho^L$. Then $z_{(i)} = \widehat{z}_{(i)} - \delta$.

  If $s_{(i-1)} = \overline{s}_{(i-1)} = \rho^L$ then $\widehat{z}_{(i-1)} = z_{(i-1)}$. Then $w_{(i)}$ is strictly between $\max\{\widehat{z}_{(i-1)}, z_{(i-1)}\}$ and $\min\{\widehat{z}_{(i)}, z_{(i)}\}$, so $\widehat{z}_{(i-1)} < w_{(i)} < z_{(i)} < \widehat{z}_{(i)} \leq w_{(i+1)}$.

  Now: $\overline{f}_{(i)}(w_{(i)}) = \widehat{y}_{(i)} + \rho^L(\widehat{z}_{(i)} - w_{(i)})$ and $\overline{f}_{(i)}(w_{(i+1)}) = \widehat{y}_{(i)} + \rho^L(\widehat{z}_{(i)} - w_{(i+1)})$. This means that the line segment between $(w_{(i)}, \kappa_{(i)})$ and $(w_{(i+1)}), \kappa_{(i+1)})$ has slope ρ^L. As $(\widehat{z}_{(i)}, \widehat{y}_{(i)})$, $z_{(i)} = \widehat{z}_{(i)} - \delta$, $y_{(i)} = \widehat{y}_{(i)} + \delta\rho^L$ and $w_{(i)} < z_{(i)} < w_{(i+1)}$, we see that $(z_{(i)}, y_{(i)})$ is also on the line segment.

  If $s_{(i-1)} > \overline{s}_{(i-1)}$ then we know that $w_{(i)} = \widehat{z}_{(i)}$ and $\kappa_{(i)} = \widehat{y}_{(i-1)}$. So $w_{(i)} < z_{(i)} < \widehat{z}_{(i-1)} \leq w_{(i+1)}$. The proof is similar to the previous case.

- If $s_{(i)} > \rho^L$ and $s_{(i+1)} = \rho^L$ then $z_{(i)} = \widehat{z}_{(i)} = z_{(i)} - \delta$. In this case, we know that $w_{(i+1)} = \widehat{z}_{(i)}$. Also $z_{(i)} \leq \widehat{z}_{(i+1)} \leq w_{(i+2)}$. Then $\overline{f}_{(i+1)}(w_{(i+2)}) = \widehat{y}_{(i+1)} + \rho^L(\widehat{z}_{(i+1)} - w_{(i+2)})$ and $\overline{f}_{(i+1)}(w_{(i+1)}) = \widehat{y}_{(i+1)} + \rho^L(\widehat{z}_{(i+1)} - w_{(i+1)})$. This means that the segment between $(w_{(i+1)}, \kappa_{(i+1)}) = (\widehat{z}_{(i)}, \widehat{y}_{(i)})$ and $(w_{(i+2)}, \kappa_{(i+2)}$ has slope with absolute value ρ^L. Given that $w_{(i+1)} = \widehat{z}_{(i)} < z_{(i)} < w_{(i+2)}$, $z_{(i)} = \widehat{z}_{(i)} + \delta$ and $y_{(i)} = \widehat{y}_{(i)} - \delta\rho^L$, we see that $(z_{(i)}, y_{(i)})$ is also on this segment.

Let us now show that for all i, j, $z_{(i)} \neq w_{(j)}$. Note that for $j \leq i - 1$,

$$w_{(j)} \leq \widehat{z}_{(i-1)} < z_{(i)},$$

and for $j \geq i + 2$,

$$z_{(i)} < \widehat{z}_{(i+1)} \leq w_{(j)}.$$

As such, we can focus on the cases where $j = i$ or $j = i + 1$.

Towards a contradiction, assume that $z_{(i)} = w_{(i)}$. We know that $w_{(i)} < \widehat{z}_{(i)}$ so this must mean that $w_{(i)} = z_{(i)} = \widehat{z}_{(i)} - \delta$.

This means that $s_{(i)} > \rho^L$. However then we have seen that: $\max\{\widehat{z}_{(i-1)}, z_{(i-1)}\} < w_{(i)} < \min\{\widehat{z}_{(i)}, z_{(i)}\}$ a contradiction.

If $z_{(i)} = w_{(i+1)}$ Then as $\widehat{z}_{(i)} \leq w_{(i+1)} \leq \widehat{z}_{(i+1)}$, we must have that either $\widehat{z}_{(i)} \leq z_{(i)}$.

If $\widehat{z}_{(i)} = z_{(i)}$, then $\widehat{z}_{(i)} = w_{(i+1)}$ so this implies that $s_{(i+1)} = \overline{s}_{(i+1)} = \rho^L$ and $s_{(i)} > \rho^L$. However this means that $\widehat{z}_{(i)} = z_{(i)} - \delta$, a contradiction.

If $\widehat{z}_{(i)} < z_{(i)}$ this means that $\widehat{z}_{(i)} = z_{(i)} - \delta$. This can only be if $s_{(i)} > \rho^L$ and $s_{(i+1)} = \rho^L$. In this case, however, we have seen that $w_{(i+1)} = \widehat{z}_{(i)} < z_{(i)}$ again a contradiction. $\square$

Above lemma shows that every point $(z_{t,s}, y_{t,s})$ will be on the (relative) interior of a segment of $\overline{f}$ with a certain slope. Let us denote this slope by $\widetilde{s}_{t,s}$. In particular, for every $z_{t,s}$ we can find an open neighbourhood around this point such that for all z in this neighbourhood $\overline{f}_z = y_{t,s} + \widetilde{s}_{t,s}(z_{t,s} - z)$.

Step 3:

The third step consists in smoothing out the function $\overline{f}$ by convolution (see for example, Chiappori and Rochet (1987)). Define the bump function:

$$\begin{aligned} g(a) &= \frac{\exp\left(\frac{-1}{1-a^2}\right)}{\int_{-1}^{1} \exp\left(\frac{-1}{1-y^2}\right) dy} \quad && \text{if } |a| < 1, \\ g(a) &= 0 && \text{if } |a| \geq 1. \end{aligned}$$

Let $\eta > 0$ such that every point $z_{t,s}$ is at least a distance η from one of the corner points of $\overline{f}(z)$ (i.e. the points $w_{(i)}$). In other words, $\overline{f}(z) = y_{t,s} + \widetilde{s}_{t,s}(z_{t,s} - z)$ for all $z \in [z_{t,s} - \eta, z_{t,s} + \eta]$.

Next, let $g_\eta(a) = \frac{1}{\eta} g(a/\eta)$. Then g_η is symmetric ($g_\eta(a) = g_\eta(-a)$), C^∞, and zero for values $|a| > \eta$. By definition, it satisfies $\int_{-\eta}^{\eta} g_\eta(\varepsilon) d\varepsilon = 1$ and by symmetry: $\int_{-\eta}^{\eta} \varepsilon g_\eta(\varepsilon) d\varepsilon = 0$. Define:

$$f^*(z) = \int_{\mathbb{R}} \overline{f}(z - \epsilon) g_\eta(\epsilon) d\epsilon.$$

This function is also C^∞. Indeed, by a change of variables $\zeta = z - \epsilon$, we can write:

$$f^*(z) = -\int_{\mathbb{R}} \overline{f}(\zeta) g_\eta(z - \zeta) d\zeta.$$

Then, C^∞ of f^* follows immediately from the fact that g_η is C^∞.

Next, given that $\overline{f}$ is non-increasing, we have that for $z' > z$:

$$f^*(z') - f^*(z) = \int_{\mathbb{R}} (\overline{f}(z - \epsilon) - \overline{f}(z' - \epsilon) g_\eta(\epsilon) d\epsilon \leq 0.$$

Thus, f^* is also non-increasing. Next, for all $t \leq T$ and $s \leq S$:

$$\begin{aligned} f^*(z_{t,s}) &= \int_{\mathbb{R}} \overline{f}(z_{t,s} - \epsilon) g_\eta(\epsilon) d\epsilon \\ &= \int_{-\eta}^{\eta} \overline{f}(z_{t,s} - \epsilon) g_\eta(\epsilon) d\epsilon, \\ &= \int_{-\eta}^{\eta} (y_{t,s} + \widetilde{s}_{t,s}\epsilon)\, g_\eta(\epsilon) d\epsilon \\ &= y_{t,s} + \widetilde{s}_{t,s} \int_{-\eta}^{\eta} \epsilon g_\eta(\epsilon) d\varepsilon = y_{t,s}. \end{aligned}$$

This shows that $f^*(z_{t,s}) = y_{t,s}$, so the function f^* interpolates the points $(z_{t,s}, y_{t,s})$. The slope of $f^*(z)$ is an average of the slopes of the function $\overline{f}(z)$, so the absolute value of this slope is in the interval $[\rho^L, \rho^U + \varepsilon]$. Given this, $f^*(z)$ is a C^∞, non-increasing function going through the points $(x_{t,s}, y_{t,s})$ whose derivative has absolute value in $[\rho^L, \rho^U + \varepsilon]$.

Fix any $\hat{x} > 0$ and consider the utility function $u(x) = \int_{\hat{x}}^{x} e^{f^*(\ln(z))} dz$. Notice that u is C^∞, concave, and strictly increasing. Also, $R(x) = -(f^*)'(\ln(x)) \in [\rho^L, \rho^U + \varepsilon]$ and the lower bound is attained. So $R^L = \rho^L$ and $R^U \leq \rho^U + \varepsilon$.

Moreover, this function rationalizes the data. To see this, define $m_t = \sum_s p_{t,s} x_{t,s}$ and consider any bundle $(x_s)_{s \leq S}$ such that $\sum_s p_{t,s} x_s \leq m_t$. By concavity and C^∞ of $u(x)$, we have:

$$\begin{aligned} \sum_s \pi_{t,s} \left(u(x_s) - u(x_{t,s})\right) &\leq \sum_s \pi_{t,s} u'(x_{t,s}) \pi_{t,s} \left(x_s - x_{t,s}\right) \\ &= \sum_s \pi_{t,s} e^{f^*(\ln(x_{t,s}))} \pi_{t,s} \left(x_s - x_{t,s}\right) \\ &= \sum_s \pi_{t,s} \exp(\delta_t) \frac{p_{t,s}}{\pi_{t,s}} (x_s - x_{t,s}) \\ &= \exp(\delta_t) \sum_s p_{t,s} (x_s - x_{t,s}) \leq 0. \end{aligned}$$

So $\mathcal{D}$ is EU rationalizable by the utility function u.

(c). The proof of this follows similar lines as the proof of part (b).

C Proof of Corollary 1

If $\mathcal{D} = (p_{t,s}, \pi_{t,s}, x_{t,s})_{t \leq T, s \leq S}$ is rationalizable by a CRRA utility function, then $R^U = R^L$, which, by Theorem 3 immediately gives that $\rho^U = \rho^L$.

For the reverse, consider the graph where we plot values of $y_{t,s} \equiv \delta_t + \ln\left(\frac{p_{t,s}}{\pi_{t,s}}\right)$ against the values of $z_{t,s} \equiv \ln(x_{t,s})$. As $\rho^U = \rho^L$, we have that all these points lie on a common line with slope $-\rho^U = -\rho^L$. Call this function $f(z)$. Denote the utility function $u(x) = \int_{\hat{x}}^{x} e^{f(\ln(z))}dz$, where $\hat{x} > 0$ is some arbitrary income level. Note that $R(x) = \rho^U = \rho^L$ for all x, so u belongs to the CRRA class of utility functions. The remainder of the proof follows the last part of Theorem 2.