Inequality, Welfare, and the Cost of Coordination Failure

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Abstract. Coordination failures arise when society gets stuck in a "bad" equilibrium when a Pareto superior one exists. How does wealth inequality affect coordination failure? This paper models a large, heterogeneous society where individuals' consumption-savings choices are influenced by investment spillovers. The *cost of coordination failure (CCF)* in this society is the welfare difference between the "good" (high investment) and the "bad" (low investment) equilibrium. We provide natural conditions under which the CCF is increasing under mean-preserving spreads in wealth inequality.

We also establish a trifurcation result in which high enough inequality creates a coordination failure where none had existed. Starting from a stable equilibrium where investment is unaffected by inequality, the equilibrium becomes unstable as inequality increases, and two other stable equilibria — a good one and a bad one — emerge. In the bad equilibrium, inequality always reduces both aggregate investment and aggregate welfare. In the good one, inequality is always investment-enhancing. Furthermore, there is a range of parameters where inequality is actually Pareto-improving in the good equilibrium. In all cases, the welfare gap (the CCF) between the two equilibria increases with inequality.

JEL Codes: C72, D3, D62.

Key Words and Phrases: Aggregative games, coordination problems, cost of coordination failure, wealth inequality.

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1 Introduction

Coordination failures commonly refer to situations where a society gets stuck in a "bad" equilibrium when a Pareto superior equilibrium exists. A large literature in economics suggests coordination failures are responsible for financial crises, market rigidities, and poverty traps.[1]

This study focusses on the relation between inequality and coordination failures. Coordination failures can occur under both high levels and low levels of inequality. Studies of poverty traps, however, often implicate within-society inequality as a factor (for instance Azariadis and Stachurski (2005) and contributors to Bowles et al. (2006)). Inequality can exacerbate coordination failures for a number of reasons. Self-fulfilling expectations can trap societies with histories of low social mobility. Individuals near the poverty threshold have little capacity to save for the future, and their responses are magnified by negative spillovers from others' reductions in savings (Barrett et al. (2019)). More generally, with positive spillovers, greater inequality can reduce overall investment when existing investment is low, and may increase overall investment when existing investment is high.

We analyze how wealth inequality impacts both "good" and "bad" equilibria in an environment where these types of coordination problems arise. We define the *cost of coordination failure (CCF)* as the resulting gap in social welfare between the two equilibria. When does greater inequality lead to a higher CCF?

To address this issue, we model a large, heterogeneous society where individuals' consumption-savings choices are influenced by investment spillovers. Each individual differs with respect to initial wealth levels, and her second period consumption depends on her saving decision in the first period and on aggregate investment. Depending on payoffs and the technology of investment returns, a coordination problem arises if two stable equilibria coexist: a Pareto superior high investment equilibrium (the "good equilibrium"), and a Pareto inferior low investment equilibrium (the "bad equilibrium").

Increases in wealth inequality are modeled as mean-preserving spreads of wealth, indexed by a one-parameter family of distributions. We focus on distributional comparative statics of equilibria with respect to changes in this inequality index. Does an increase in the index lead to a larger cost of coordination failure?

Our answer makes use of a measure of *inequality tolerance*. While mathematically equivalent to the traditional Arrow-Pratt measure of risk tolerance, the notion of inequality tolerance embodies a conceptually different idea. In the present model, individuals face no decision risk. Each agent makes her consumption-savings choice knowing her own wealth and facing no second period shock. The mean preserving spread represents increased dispersion across different individuals rather than increased dispersion in future states of nature faced by a given individual. Inequality tolerance captures the willingness

[1]See Section 2 for references.

of the social planner to tolerate a mean-preserving spread across individuals when each individual can make her savings decision later on, *after* her new position in society is realized?

Our main result shows that increasing wealth inequality raises the cost of coordination failure under fairly broad conditions. Specifically, inequality increases the CCF whenever:

(1) The discounted return to investment in the good (bad) equilibrium lies above (below) one.

(2) Inequality tolerance is increasing and strictly convex in one's wealth.

In other words, under Conditions (1)-(2), the welfare difference between the good and bad equilibrium grows as wealth inequality increases.

Condition (1) is a separating condition. It assumes the two coordination equilibria are on opposite sides of a divide that determines whether returns to first or second period consumption are higher at the margin. This condition is typically needed for a coordination problem to arise in the first place. As for Condition (2), the assumption of increasing inequality tolerance is equivalent to an assumption of DARA (decreasing absolute risk aversion). Convexity of inequality tolerance in Condition (2) means the marginal willingness to tolerate inequality is increasing in one's wealth. In its weak form (weak convexity) it is satisfied by CRRA (constant relative risk aversion) payoff functions. In these cases inequality tolerance is linear in wealth.[2] Yet, the strict convexity assumed in our results is, in a sense, more robust. It is satisfied, for instance, by convex combinations of arbitrary CRRA utility functions, including CES and log functions.

To establish the main result, we first show that Conditions (1) and (2) imply an individual's savings function is convex in wealth in the good equilibrium and is concave in the bad one. This implies that more inequality increases aggregate investment in the good equilibrium and decreases it in the bad one.

Yet, while aggregate investment has a positive influence on welfare, there are countervailing influences. We decompose the change in CCF into two distinct adjustments. The first adjustment looks at changes in CCF holding the aggregate investment spillover constant. This effect captures the welfare difference between good and bad equilibria when each individual views her new wealth position as idiosyncratic. She adjusts her savings assuming aggregate investment remains fixed. We dub this the *Atkinson wedge* because its calculation is based on the equivalence, established in Atkinson (1970), between

[2] Carroll and Kimball (1996) refer to the case where the Arrow-Pratt measure is linear as *hyperbolic absolute risk aversion (HARA)*, and use HARA to establish concavity consumption functions when returns are stochastic. Jensen (2018) later establishes a link between HARA and quasi-concave differences. The latter concept is introduced by Jensen to produce a tractable framework for distributional comparative statics. We later discuss the relation between our results and these.

increasing Lorenz inequality in the population and welfare losses from second order stochastic increases in individual risk. This downside affects both equilibria. However, downside losses from a mean-preserving spread are greater (smaller) for low wealth savers than are the gains from high wealth savers in the bad (good) equilibrium. Thus, the "pure Atkinson loss" in welfare from increased dispersion is larger in the bad than in the good equilibrium, and so the Atkinson wedge is positive.

The second adjustment, the *spillover wedge*, refers to relative differences across the two equilibria in spillover effects holding constant the welfare losses from pure wealth dispersion. The spillover effect on CCF depends on how individuals at different ends of the wealth distribution react to, and determine, changes in the aggregate return. Intuitively, in a coordination problem, the different equilibria arise from different self-fulfilling expectations. In this case, the self-fulfilling expectations concern the size of the spillover. If the spillover is expected to be small, individual savings curves are concave. A mean-preserving spread then disproportionately alters the behavior of low-end savers whose wealth levels have decreased due to the spread. If, however, the spillover is expected to be large, individual savings curves are convex. A mean-preserving spread then disproportionately alters the behavior of high-end savers whose wealth levels have increased. Looking at the difference between equilibria, the spillover wedge is also positive.

A second set of results show that high enough inequality can produce a potential coordination failure where none existed before. To elaborate, consider a benchmark consumption-savings game where individual and aggregate savings are strategic complements. We show that for low levels of inequality this game admits a "neutral" equilibrium — a unique, stable equilibrium level of investment which is locally invariant to changes in inequality. We then prove a trifurcation ("pitchfork") result as inequality increases. Starting from the neutral equilibrium, as inequality increases it eventually reaches a threshold at which the equilibrium set trifurcates. Two stable, Pareto ranked equilibria emerge on opposite sides of the neutral equilibrium, and the neutral equilibrium itself becomes unstable.

In the bad equilibrium higher inequality is always welfare-reducing. However, in the good equilibrium welfare temporarily increases in a range above the threshold. Indeed, there is a range above the threshold where increases in inequality are actually Pareto-improving. In this range, positive spillovers from wealthy savers outweigh the negative Atkinson effects of inequality. Nevertheless, at high enough levels of inequality, welfare is reduced in the good equilibrium as well. This means that, while the welfare-maximizing level of inequality is the lowest possible in the bad equilibrium, there is a positive, welfare-maximizing level of inequality in the good equilibrium. The cost of coordination failure therefore widens, but does so in a way that complicates the standard narrative about welfare and inequality.

Putting everything together: (i) at a high enough level of inequality a coordination problem emerges; (ii) increasing inequality is always harmful in the bad equilibrium; (iii) increasing inequality is beneficial at first in the good equilibrium then becomes harmful again; (iv) increasing inequality increases the welfare difference between the two.

The results address an ongoing debate in macroeconomics and political economy regarding whether inequality primarily promotes dynamism and investment or instead generates macroeconomic fragility and underperformance. On one side, the Schumpeterian tradition emphasizes that some degree of dispersion in wealth and income boosts incentives for entrepreneurship, innovation, and accumulation (see, e.g., Aghion and Howitt (1992); Aghion et al. (2019)). On the other side, Piketty (2014) and Stiglitz (2012) argue that inequality may suppress aggregate demand, weaken social cohesion, amplify economic insecurity, and ultimately reduce investment and welfare. Existing contributions typically frame these perspectives as competing hypotheses about the average impact of inequality on economic performance. However, the empirical evidence remains inconclusive, with different studies finding positive (Forbes (2000); Partridge (1997)), negative (Persson and Tabellini (1994); Alesina and Rodrik (1994)), or highly nonlinear relationships Banerjee and Duflo (2003); Barro (2000)) between inequality, investment, and growth.

This divergence of empirical findings becomes less surprising through the lens of our analysis, which makes a step towards reconciling the two opposing views on inequality. We stress that inequality is not intrinsically good or bad: its consequences may depend critically on equilibrium selection and coordination. In the "good" equilibrium, dispersion in wealth stimulates aggregate investment and may even increase welfare, which resonates with the Schumpeterian view. In the "bad" equilibrium, wealth disparities unambiguously reduce investment and welfare, in line with the fragility and underperformance mechanisms. Thus, inequality increases the returns to successful coordination while simultaneously increasing the cost of coordination failure.

There are also practical implications. On the one hand, an increasing CCF means more unequal societies may face more severe poverty traps than more equal ones. This is consistent with empirical findings (for instance, Flug et al. (1998); Adato et al. (2006); Balboni et al. (2022)). On the other hand, an increasing cost of coordination failure can push societies to solve their coordination problems. It increases the reward for social, cultural, and political changes that move society toward the preferred equilibrium. Experimental evidence by Bigoni et al. (2024) suggests this is the case. In either case, the debate on whether inequality is harmful or beneficial should account for equilibrium selection.

Up next, Section 2 briefly reviews the literature on coordination failures and inequality. Section 3 introduces the general investment model with potentially multiple equilibria and introduces an external shock in the form of a mean-preserving increase in wealth inequality. Section 4 states the main result relating inequality to the cost of coordination failure. Section 5 proves the "pitchfork" result. Starting from a stable equilibrium, as inequality increases, the equilibrium set eventually trifurcates. Two stable, Pareto ranked equilibria and one unstable equilibrium emerge. Inequality reduces welfare in the bad equilibrium but can raises welfare in the good one for a range of parameters.

Section 6 constructs a parametric model to illustrate the trade offs explicitly. Section 7 revisits a discussion of key assumptions and implications. Section 8 contains all the proofs.

2 Models of Coordination Failure and Inequality

A large literature in economics suggests coordination failures are responsible for financial crises, market rigidities, and poverty traps. A partial account includes bank runs and financial collapse (Diamond and Dybvig (1983); Bryant (1980); Lorenzoni (2008)) and Marshall 1998), labor market rigidities (Dybvig and Jaynes (1979, 1980); Michaillat and Saez (2015); Howitt (1985, 1988); Howitt and McAfee (1987); Bouvard and de Motta (2021), and Rodrik (1996)), locational choice (Adsera and Ray (1998)), social stratification (Matsuyama (2006)), social preference for fairness (Alesina and Angeletos (2005)), investment spillovers, credit market rigidities, and increasing returns (Skiba (1978); Cooper and John (1988); Galor and Zeira (1993); Matsuyama (1991); Kiyotaki and Moore (1997), and more recently, Moll (2014)), and relatedly, poverty traps (Murphy et al. (1989); Azariadis and Stachurski (2005); Azariadis and Drazen (1990); Bowles et al. (2006) and Bryan et al. (2014) (for field evidence).[3]

This paper follows the long tradition of studies on investment spillovers as a natural source of coordination failure (e.g., Skiba (1978); Murphy et al. (1989); Azariadis and Stachurski (2005); Azariadis and Drazen (1990)). Strategic complementarities inherent in these spillovers often produce multiple equilibria including ones with suboptimal investment (Van Huyck et al. (1990); Angeletos et al. (2006)). By design, we place coordination failures in a simple, intuitive framework. The results require neither labor-market nor credit-market rigidities nor social preference for fairness assumed in the literature.

Theoretical foundations of the effects of inequality on social welfare date at least back to Atkinson (1970). We focus on distributional comparative statics of aggregate investment. Comprehensive work of Jensen (2018) establishes under very general conditions the monotone link between inequality and aggregate outcomes like investment.[4] Jensen introduces the notion of quasi-concave/quasi-convex differences in payoffs and shows it implies global concavity/convexity of the best reply functions in large games. Unfortunately, our study cannot fully utilize these global properties because the conditions that yield multiple equilibria will entail best responses that are not globally concave or convex. Our study focuses on coordination problems, and the curvature of the best response will depend on which equilibrium the society is in.

[3] Hoff (2000) provides an excellent survey of older work.
[4] See also Acemoglu and Jensen (2015); Camacho et al. (2018); Anderson and de Palma (2024).

3 The Model

We model a class of large heterogeneous societies, all populated by a unit-mass of individuals. Individuals in each society differ by wealth $w > 0$. The wealth distribution is summarized by differentiable density $g(w)$ with finite mean $\mathbb{E}[w] = \int wg(w)dw < \infty$.

3.1 Individual Decisions

Each individual in this society faces a standard consumption-savings decision. Each receives a flow payoff over two time periods: $t = 0$ (the present) and $t = 1$ (the future). Her lifetime utility function describes preferences over possible consumption paths (c_0, c_1) according to

$$\mathcal{U}(c_0, c_1) \; = \; u(c_0) + \beta u(c_1). \tag{1}$$

Payoffs are identical across consumers. In Equation (1), $u(\cdot)$ is the flow payoff over consumption at each date, and $\beta \in (0,1)$ is the discount factor. Assume u is strictly concave, increasing, and smooth (at least up to the fourth derivative), and satisfies the usual transversality conditions, $u'(c) \to \infty$ as $c \to 0$ and $u'(c) \to 0$ as $c \to \infty$.

In the initial period, an individual with wealth w chooses a level of savings x that maximizes

$$\mathcal{U}(w - x, \, Rx) \; = \; u(w - x) \; + \; \beta u(Rx). \tag{2}$$

given a (forecasted) per unit return of $R > 0$ in the second period. From the individual's point of view, the forecasted return R is exogenous.

Let $s(w, R)$ denote a solution to (2). The savings function satisfies the first order condition

$$u'(w - x) \; = \; \beta R \, u'(Rx) \tag{3}$$

Given the assumptions on u, the solution to (3) is an interior solution $0 < s(w, R) < w$ which is continuous and increasing w, and $s_w(w, R) < 1$ for all R. That is, first period's consumption is not an inferior good.

All our results will make use of the structure of *inequality tolerance*,

$$T_u(c) \; \equiv \; -\frac{u'(c)}{u''(c)} \tag{4}$$

given payoff function u and consumption c. As the name indicates, T_u measures a society's willingness to tolerate inequality. Inequality tolerance is mathematically identical to the more familiar Arrow-Pratt measure of risk tolerance, the reciprocal of the measure of absolute risk aversion.

However, inequality tolerance captures a conceptually different idea. In this model, individuals face no decision risk. Each agent makes her consumption-savings choice knowing her own wealth and facing no second period shock. Hence, the mean preserving spread represents increased dispersion in the population rather than increased dispersion in future states of nature faced by a given individual. Inequality tolerance captures the willingness of a social planner to tolerate a mean-preserving spread across individuals when each individual can make her savings decision later on, after her new wealth position is realized.

Lemma 1. *Given flow payoff u, then the following statements are equivalent.*

1. *The inequality tolerance $T_u(\cdot)$ is strictly convex in c.*
2. *The savings function $s(\cdot, R)$ is (strictly) convex in w if and only if $\beta R (>) \geq 1$.*

The proof of this and all results are in Section 8 at the end.[5] The Lemma is a workhorse used in all subsequent results. As a key assumption, the interpretation of a convex T_u is worth unpacking. This property is equivalent to assuming $T_u'(c)$ increasing in c. Using c as a stand-in for wealth, when $T_u'(c) > 0$, wealthier individuals have higher inequality tolerance. If, in addition, T_u is convex, inequality tolerance as wealth increases is increasing at an increasing rate.

An alternative interpretation makes use of Kimball's notion of *prudence* (Kimball (1990)). Letting $r_u(c) \equiv (T_u(c))^{-1}$ denote the standard Arrow-Pratt curvature, then $T_u'(c) = \frac{r_{u'}(c)}{r_u(c)} - 1$. Kimball's prudence measures the precautionary savings motive, independently of insurance motives. The Arrow-Pratt curvature r_u in the denominator measures the insurance motive. The ratio $\frac{r_{u'}(c)}{r_u(c)} - 1$ is therefore a measure of the strength of the pure precautionary motive relative to the insurance motive. Evidently, the higher this strength, the higher one's tolerance for risk.[6]

Lemma 1 entails different assumptions than those needed if risk were present at the time one makes a decision. By contrast, Carroll and Kimball (1996) combine decision risk with the assumption $T_u''(c) = 0$ to prove concavity of the consumption function in a consumption-savings model. The assumption $T_u''(c) = 0$ means risk/inequality tolerance is linear in wealth, a property they refer to as hyperbolic average risk aversion (HARA). Jensen (2018) shows in the same stochastic model that a simple condition, quasi-concave

[5] In an External Appendix, we prove an even stronger result under smoothness of u, namely:

$$\frac{\partial^2 s}{\partial w^2}(w, R) \gtreqless 0 \iff T_u''(\cdot)\left(R - \frac{1}{\beta}\right) \gtreqless 0.$$

Thus, we can derive properties on savings when T_u is concave. We restrict attention to the case where T_u is convex, i.e., $T_u''(c) > 0$ since this seems the more natural case.

[6] Wilson (1968) refers to this ratio as a measure of *cautiousness*, where higher values indicate lower cautiousness. In the language of Wilson, we assume wealthier individuals are less cautious, and increasingly so.

differences, implies HARA and generates a concave consumption function. Their results stem from decision risk. Without decision risk, HARA utility, including CES and log, are linear in wealth. With $T_u''(c) > 0$, consumption can be either convex or concave in wealth depending on whether a simple discounted returns threshold is exceeded.

Strict convexity of T_u is actually a broad condition. It is satisfied, for instance, by any strict, convex combination of distinct CRRA functions. More precisely:

Remark. *Inequality tolerance $T_u(c)$ is strictly increasing and strictly convex if*

$$u(c) = \sum_{j=1}^{J} \delta_i \frac{c^{1-\gamma_j} - 1}{1 - \gamma_j}, \tag{5}$$

where $\gamma_j, \delta_j \in (0, 1)$ for all j, with $\gamma_k \neq \gamma_j$ for at least one k and j pair, and $\sum_{j=1}^{J} \delta_j = 1$.[7]

Clearly, CES payoffs are limits where $\delta_j = 1$ for some j. Hence, strict convexity of T_u is achieved by a robust set of perturbations of the CES case.

3.2 Equilibrium Aggregate Investment

Given each individual's savings function $s(w, R)$, the aggregate savings function is

$$S(R) \equiv \mathbb{E}[s(w, R)] = \int_{\underline{w}}^{\overline{w}} s(w, R) g(w) dw. \tag{6}$$

The individual's savings problem is embedded in a large society with investment spillovers. The return R is determined by a return function $\mathcal{R}(X) = R$ that varies with aggregate investment level X. We assume for now that $\mathcal{R}$ is weakly increasing, bounded, and twice differentiable almost everywhere, satisfies $\mathcal{R}(0) > 0$, and $\lim_{X\to\infty} \frac{d}{dX} \log(\mathcal{R}(X)) = 0$. This last assumption is needed to ensure that returns do not grow too fast, a necessary condition for existence of equilibria.

To close the model, a Nash equilibrium (simply "equilibrium") is defined as a mapping x^* with $x^*(w) = s(w, \mathcal{R}(X^*))$ where X^* is aggregate investment:

$$X^* \equiv \int_0^\infty x^*(w) g(w) dw. \tag{7}$$

By definition, the equilibrium aggregate investment X^* satisfies the fixed-point condition $X^* = S(\mathcal{R}(X^*))$, aggregate savings equals aggregate investment. An equilibrium x^* is *stable* if $S'(\mathcal{R}(X^*))\mathcal{R}'(X^*) < 1$.

This setup describes a large aggregative game in the sense of Jensen (2016). The spillover from others' decisions enter an individual's payoff only via aggregate investment

[7] The proof is in the appendix.

X. Other attributes of the wealth distribution do not directly affect payoffs. Because each individual has negligible influence on X, she treats the return $R = \mathcal{R}(X)$ as given.

Let $R^* = \mathcal{R}(X^*)$ denote the equilibrium return. An individual's payoff in equilibrium $x^*(w) = s(w, R^*)$ is

$$v(w, R^*) \;=\; u\left(w - s(w, R^*)\right) \;+\; \beta u\left(R^* s(w, R^*)\right). \tag{8}$$

We focus on social welfare with equal treatment in equilibrium, as given by

$$V(R^*) \;=\; \int v(w, R^*) g(w) dw \tag{9}$$

3.3 Cost of Coordination Failure

The consumption-savings game described so far is characterized by a profile $(u, g, \mathcal{R}, \beta)$ consisting of the flow payoff function u, the density g of the wealth distribution, the return function $\mathcal{R}$, and the discount factor β.

The consumption-savings game $(u, g, \mathcal{R}, \beta)$ *admits a coordination problem* if there are two stable equilibria, $\overline{x}^*$ and $\underline{x}^*$ with corresponding aggregate investment levels $\overline{X}^* > \underline{X}^*$ and associated returns $\overline{R}^* > \underline{R}^*$. The high investment equilibrium $\overline{X}^*$ Pareto dominates the low investment equilibrium $\underline{X}^*$, and so $V(\overline{R}^*) \;>\; V(\underline{R}^*)$.

For brevity, we refer to $\overline{X}^*$ as the "good equilibrium" and $\underline{X}^*$ as the "bad equilibrium." Later on, we find sufficient conditions that ensure the existence a coordination problem. For now, we assume a coordination problem exists and work out its implications. Later on (Section 5) we posit structure that ensures its existence.

The *cost of coordination failure (CCF)* is defined as the difference in social welfare between the two equilibria, i.e.,

$$\Delta V \;=\; V(\overline{R}^*) \;-\; V(\underline{R}^*) \tag{10}$$

The CCF is the welfare loss of a coordination failure when compared to the Pareto dominant equilibrium. Our interest is in the comparative statics of ΔV, and in particular, how ΔV changes as wealth inequality increases.

3.4 Increasing Wealth Inequality

This section considers the consequences of an increase in wealth inequality. We posit a class of otherwise identical societies ordered by mean-preserving spreads. Formally, a homotopic class is a collection $\{g(w|\theta) : \;\; \theta \in \mathbb{R}\}$ of wealth distributions where θ is a linearly ordered measure of inequality/dispersion in the density $g(w|\theta)$ such that increases in θ represents mean-preserving spreads of wealth.[8] Thus, $\mathbb{E}[w|\,\theta] = \mathbb{E}[w]$ for all θ, and wealth inequality is greater the higher is θ.

An obvious candidate for θ is the variance of g. However, it is not essential that θ correspond to a specific moment. We assume the variance $\mathbb{E}[w^2|\theta]$ is finite for fixed θ, but $\lim_{\theta\to\infty} \mathbb{E}[w^2|\theta] = \infty$, that is, the variance increases without bound as θ becomes large.

Individual and aggregate investment and returns in equilibrium are expressed now as functions of θ, namely, write $x^*(w,\theta)$, $X^*(\theta)$, and $R^*(\theta)$, respectively. Social welfare is expressed as

$$V(R^*(\theta),\ \theta) \ = \ \int v(w, R^*(\theta))g(w|\theta)dw. \tag{11}$$

4 Cost of Coordination Failure and Wealth Inequality

Returning to the coordination problem, denote the equilibrium returns in each equilibrium by

$$\overline{R}^*(\theta) \ \equiv \ \mathcal{R}(\overline{X}^*(\theta)) \ \text{ and } \ \underline{R}^*(\theta) \ \equiv \ \mathcal{R}(\underline{X}^*(\theta)), \tag{12}$$

and so the cost of coordination failure (CCF) defined in (10) is expressed as

$$\Delta V(\theta) \ = \ V(\overline{R}^*(\theta),\theta) \ - \ V(\underline{R}^*(\theta),\theta).$$

For comparative statics in θ, totally differentiate $\Delta V(\theta)$ with respect to θ.[9] This yields the decomposition

$$\begin{aligned} \frac{d}{d\theta}(\Delta V(\theta)) \ &= \ \underbrace{\int_0^\infty v(w,\overline{R}^*(\theta)g_\theta(w|\theta)dw \ - \ \int_0^\infty v(w,\underline{R}^*(\theta))g_\theta(w|\theta)dw}_{\textit{Atkinson Wedge}} \\ &+ \ \underbrace{\int_0^\infty \left(\frac{\partial v(w,\overline{R}^*(\theta))}{\partial R}\frac{d\overline{R}^*(\theta)}{d\theta} \ - \ \frac{\partial v(w,\underline{R}^*(\theta))}{\partial R}\frac{d\underline{R}^*(\theta)}{d\theta} \right) g(w|\theta)dw}_{\textit{Spillover Wedge}} \end{aligned} \tag{13}$$

[8]Formally, if for two distinct indices, $\theta < \hat{\theta}$, $g(\cdot|\hat{\theta})$ is a mean preserving spread of $g(\cdot|\theta)$ if

$$\int_0^{\hat{w}}\int_0^{\tilde{w}} g(w|\hat{\theta})dw\,d\tilde{w} \ > \ \int_0^{\hat{w}}\int_{\underline{w}}^{\tilde{w}} g(w|\theta)dw\,d\tilde{w}$$

for all $\hat{w} < \overline{w}$ and

$$\int_0^{\infty}\int_0^{\tilde{w}} g(w|\hat{\theta})dw\,d\tilde{w} \ = \ \int_0^{\infty}\int_0^{\tilde{w}} g(w|\theta)dw\,d\tilde{w}$$

See, for instance, Diamond and Stiglitz (1974).

[9]The explicit derivation is in the Section 8 appendix.

The CCF is increasing if $\frac{d}{d\theta}(\Delta V(\theta)) > 0$. The right-hand side (13) decomposes the effects of a change in θ into two bracketed terms. We dub the first bracketed term in (13) the *Atkinson wedge* since it isolates the effects described by Atkinson (1970) who constructed measures of inequality aversion by a social planner based on second order stochastic increases in risk to a representative agent. This is, by now, well known, and it applies to both the good and the bad equilibria. The Atkinson wedge describes the difference in welfare reductions between the two equilibria due solely to spreading out individual types, holding fixed the spillover effects.

Lemma 2. *Suppose the consumption-savings game $(u, g, \mathcal{R}, \beta)$ admits a coordination problem at some θ, and u satisfies $T_u'(c) > 0$ (increasing inequality tolerance). Then the Atkinson Wedge, evaluated at θ, is positive.*

The simple proof shows that Atkinson's notion of inequality aversion is lower when returns to savings are high than when it is low.

The second bracketed term in (13) is dubbed the *Spillover wedge* since it comes from the effects of aggregate investment. Specifically, as inequality increases, individuals adjust not only to their own positional change, but also to the external change in the return which, itself, adjusts to changes in the aggregate investment. This difference can also be signed under certain conditions, stated as follows.

Lemma 3. *Suppose the consumption-savings game $(u, g, \mathcal{R}, \beta)$ admits a coordination problem at some θ. Suppose $T_u''(c) > 0$ (inequality tolerance is convex). Suppose the equilibrium returns satisfy*

$$\underline{R}^*(\theta) < \frac{1}{\beta} < \overline{R}^*(\theta). \tag{14}$$

Then $\frac{d\overline{X}^*(\theta)}{d\theta} > 0$ and $\frac{d\underline{X}^*(\theta)}{d\theta} < 0$, and so $\frac{d\overline{R}^*(\theta)}{d\theta} > 0$ and $\frac{d\underline{R}^*(\theta)}{d\theta} < 0$. Thus, the Spillover Wedge, evaluated at θ, is positive.

Condition (14) distinguishes the two equilibria by requiring their discounted returns lie on opposite sides of unity. The good equilibria has the higher return. The statements $\frac{d\overline{X}^*(\theta)}{d\theta} > 0$ and $\frac{d\underline{X}^*(\theta)}{d\theta} < 0$ mean that equilibrium aggregate investments in the two equilibria respond in opposite directions to an increase in wealth inequality. Investment and returns in the high investment equilibrium increase while investment and returns in the low investment equilibrium decrease. From this we can conclude that the spillover effect is positive. Namely, an increase in wealth inequality reduces the portion of CCF due purely to spillovers.

Intuitively, in a coordination problem, the different equilibria arise from differing self-fulfilling expectations. In this case, the self-fulfilling expectations concern the size of the spillover. If the spillover (and associated return) is expected to be small, individual savings curves are concave by Lemma 1. A mean-preserving spread then disproportionately alters the behavior of low-end savers whose wealth levels declined due to the spread. If, however, the spillover is expected to be large, individual savings curves are convex. A mean-preserving spread then disproportionately alters the behavior of high-end savers whose wealth levels rose.[10]

The result has implications for societies on different ends of the coordination problem. Consider two societies, one in the good equilibrium and one in the bad. By Lemma 3, $d\overline{X}^*/d\theta > 0$ and $d\underline{X}^*/d\theta < 0$ so that increases in inequality move aggregate investment in opposite directions. Now consider a dynamic version of this society where an individual lives 1 period, has altruistic preferences for one's child, then leaves bequest $s(w, R, \theta)$ to the child. Given the spillover from the return technology, it follows that the inequality shock produces higher growth next period for a society in the good equilibrium; lower growth next period for a society in the bad one.

The differences between two such societies extend to policy. A standard argument in favor of inequality-reducing redistributive taxation schemes is that the winners/losers under such schemes are those whose wealth is below/above the mean $\mathbb{E}[w]$. Since the median wealth is typically below the mean wealth, that means the majority gains from redistributive taxation. In our setting a redistributive taxation scheme can be captured as a reduction in inequality index θ. In the bad equilibrium this leads to an increase in the return $\underline{R}(\theta)$. Since the mean wealth $\mathbb{E}[w]$ is preserved by the taxation scheme, an individual with mean outcome enjoys strictly positive gains from redistributive taxation which come from an increase in investment returns. However, this is no longer the case in the good equilibrium, because a reduction in θ leads to a reduction in $\overline{R}(\theta)$. Hence the share of losers/winners is greater/smaller than the share of those whose wealth is above/below the mean. This suggests the standard arguments in favor of reducing inequality should be used with caution.

Together Lemmas 2 and 3 imply

Theorem 1. *Suppose the game $(u, g, \mathcal{R}, \beta)$ admits a coordination problem at θ. Suppose*

(i) The equilibrium separation property in (14) holds.

(ii) $T_u'(c) > 0$ (inequality tolerance is increasing in c).

(iii) $T_u''(c) > 0$ (inequality tolerance is convex).

Then the CCF is increasing in θ. Furthermore, $\overline{X}^*(\theta)$ is increasing in θ, and $\underline{X}^*(\theta)$ is decreasing in θ.

As per Lemma 3, aggregate investment moves in opposite directions, increasing in the good equilibrium and decreasing in the bad one. Social welfare in the bad equilibrium is always decreasing in inequality. Individuals are worse off on average, though there could

[10] Kaldor (1955) made a related argument long ago. The Lemma shows this line of reasoning is sensitive to equilibrium selection.

be individual winners. The overall welfare effect of inequality on the good equilibrium is ambiguous. The Atkinson effect is negative. The spillover wedge from improved aggregate performance is positive. Later on, we identify natural conditions under which the spillover gain outweighs the Atkinson loss. The Theorem shows, however, that even when the overall effect in the good equilibrium is negative, the welfare loss is larger in the bad equilibrium. A leading example developed in Section 6 satisfies all the conditions of Theorem 1.

An analogue of the Theorem exists for decreasing CCF. Namely, CCF declines with inequality whenever u satisfies decreasing inequality tolerance and has a concave tolerance measure. To prove this, one needs only reverse the inequalities everywhere. We do not emphasize the result since decreasing inequality tolerance, the equivalent to increasing average risk aversion, is difficult to justify.

The results so far already presume the existence of multiple equilibria and potential coordination failure. In the next section, we address the existence of coordination problems and link it to the level of inequality.

5 Inequality Creates Coordination Problems: A Pitchfork Result

This section shows that increased inequality can itself be the source of coordination failure. We start from a benchmark, stable equilibrium in which local changes in inequality have no aggregate effect. This benchmark is referred to as a *neutral equilibrium*. Formally, a consumption-savings game (u, g, R, β) admits a neutral equilibrium at θ if the aggregate investment in this equilibrium satisfies $\frac{dX^*(\theta)}{d\theta} = 0$. In words, a neutral equilibrium is an equilibrium where aggregate investment is locally invariant to changes in inequality.

Lemma 4. *A consumption-savings game* $(u, g, \mathcal{R}, \beta)$ *admits a* neutral equilibrium *at every* θ *if and only if*

$$\mathcal{R}\left(\frac{\beta}{1+\beta}\mathbb{E}\left[w\right]\right) = \frac{1}{\beta}. \tag{15}$$

Aggregate investment in the neutral equilibrium is $X^*(\theta) = \frac{\beta}{1+\beta}\mathbb{E}\left[w\right]$.

Figure 1 displays a return technology that admits a neutral equilibrium. At value $X^*(\theta) = \frac{\beta}{1+\beta}\mathbb{E}\left[w\right]$, the discounted return is unity.

Theorem 2. *Suppose the consumption-savings game* $(u, g, \mathcal{R}, \beta)$ *admits a neutral equilibrium at each* θ. *Suppose*

(i) $T_u'(c) > 0$,

(ii) $T_u''(c) > 0$, *and*

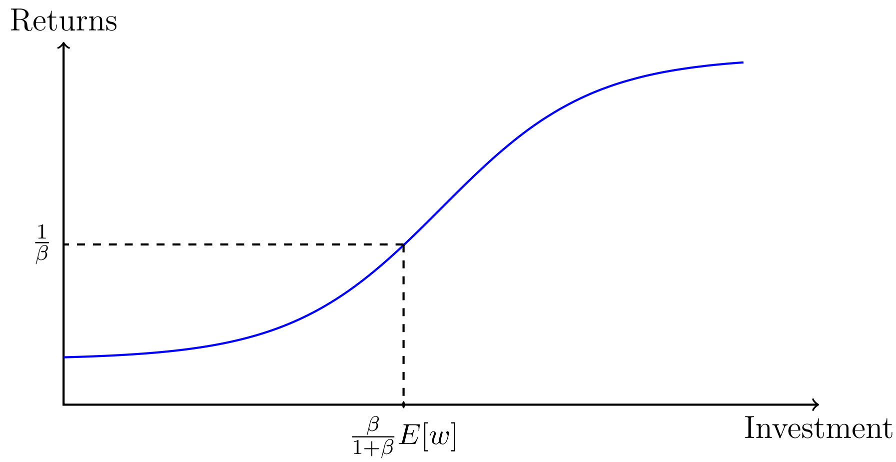

Figure 1: neutral equilibrium

(iii) $T_u(c) > c$.

Then there exists a level $\widehat{\theta}$ *of inequality such that*

(1) the neutral equilibrium is stable whenever $\theta \leq \widehat{\theta}$*, and unstable if* $\theta > \widehat{\theta}$*, and*

(2) whenever $\theta > \widehat{\theta}$*, the game admits a coordination problem. Namely, two new stable, Pareto ranked equilibria,* $\overline{X}^*(\theta)$ *and* $\underline{X}^*(\theta)$*, appear. These new equilibria generate returns* $\overline{R}^*(\theta)$ *and* $\underline{R}^*(\theta)$*, resp., and satisfy*

(a) $\overline{X}^*(\theta) \;>\; \dfrac{\beta}{1+\beta}\mathbb{E}\left[w\right] > \underline{X}^*(\theta)$,

(b) $\lim\limits_{\theta\to\widehat{\theta}} \overline{X}^*(\theta) \;=\; \dfrac{\beta}{1+\beta}\mathbb{E}\left[w\right] \;=\; \lim\limits_{\theta\to\widehat{\theta}} \underline{X}^*(\theta)$,

(c) $\overline{R}^*(\theta) \;>\; \dfrac{1}{\beta} > \underline{R}^*(\theta)$*, and*

(d) $\lim\limits_{\theta\to\widehat{\theta}} \overline{R}^*(\theta) \;=\; \dfrac{1}{\beta} = \lim\limits_{\theta\to\widehat{\theta}} \underline{R}^*(\theta)$.

When $\theta > \widehat{\theta}$, the game displays the coordination problem assumed in the previous section. The theorem requires an additional assumption in (iii): $T_u(c) > c$. This is a necessary and sufficient condition for the consumption-savings game to exhibit strategic complements, a common feature of coordination problems.[11]

Conditions (2a)-(2d) describe a trifurcation in reverse: as θ moves toward $\widehat{\theta}$, the trifurcation collapses back into the stable neutral equilibrium - an equilibrium where

[11] The condition $T_u(c) > c$ is also equivalent to assuming inelastic relative risk aversion.

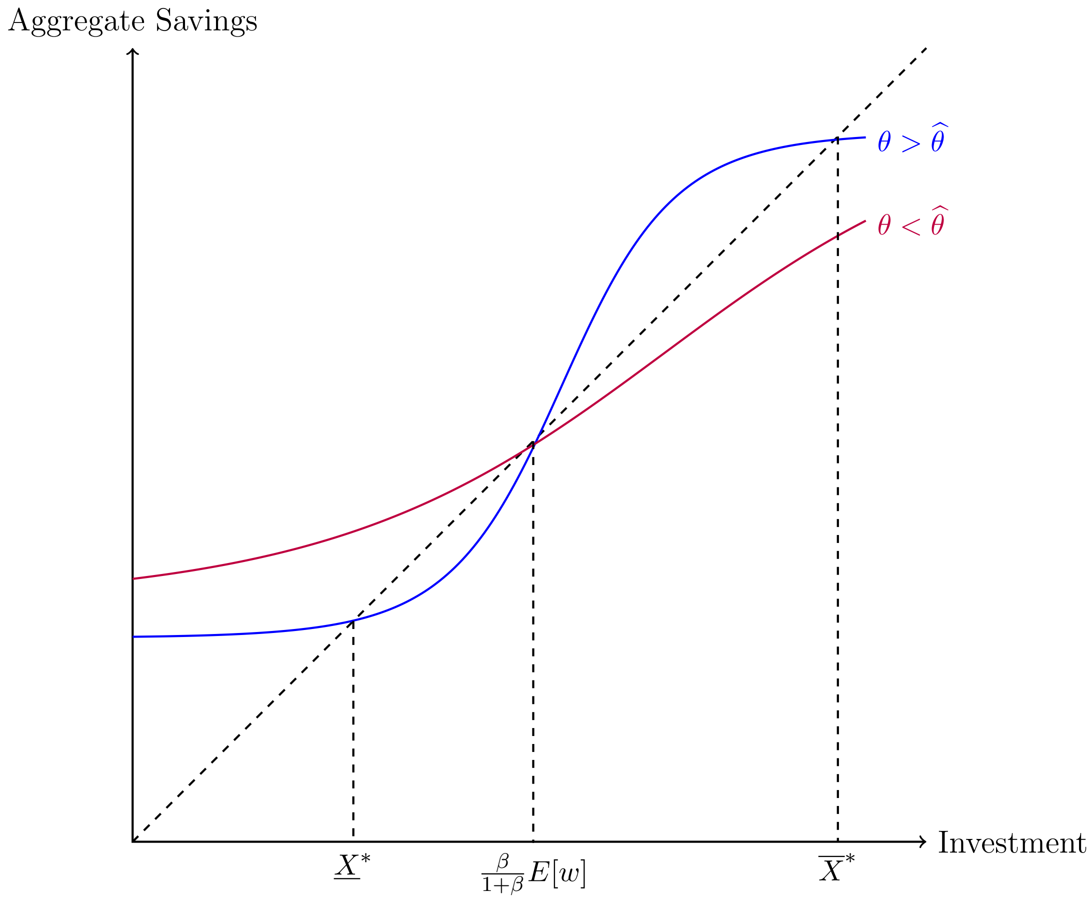

Figure 2: Fixed point maps: (i) Stable, neutral equilibrium when $\theta < \widehat{\theta}$. (ii) Coordination problem when $\theta > \widehat{\theta}$

changes in inequality have no aggregate effect. Hence, high enough inequality produces a coordination problem, and from Theorem 1, we know that CCF increases with θ in this range.

Figure 2 illustrates an aggregate savings function and its fixed points for a society with inequality $\theta < \widehat{\theta}$ and another for a society with inequality $\theta > \widehat{\theta}$. In the first case, there is a unique, neutral, stable equilibrium. In the second, the neutral equilibrium is unstable and two new, Pareto ranked, equilibria $\underline{X}^*$ and $\overline{X}^*$ emerge.[12] Though not shown in this figure, when $\theta = \widehat{\theta}$, the fixed point map is tangent to the 45° line at neutral investment $X = \mathbb{E}[w]\,\beta/(1+\beta)$.

Theorem 3. *Let $(u, g, \mathcal{R}, \beta)$ satisfy the same conditions of Theorem 2. Given threshold $\widehat{\theta}$ at which the trifurcation occurs, there are two values $\overline{\theta}$ and $\widetilde{\theta}$ with $\widetilde{\theta} > \overline{\theta} > \widehat{\theta}$, such that*

(i) if $\theta \in [\widehat{\theta}, \widetilde{\theta}]$ then inequality is welfare enhancing in the good equilibrium. Specifically,

[12] A numerical example is computed and displayed in Section 6.

$\overline{V}^*(\theta)$ is increasing in θ.

(ii) If $\theta \in \left(\widehat{\theta}, \overline{\theta}\right)$ then inequality is Pareto improving in the good equilibrium and Pareto-worsening in the bad equilibrium.

In the interval $[\widehat{\theta}, \widetilde{\theta}]$, inequality is improving welfare in the good equilibrium. An earlier result already showed that inequality always reduces welfare in the bad equilibrium. Stronger still, within a subinterval $\left(\widehat{\theta}, \overline{\theta}\right)$ of the original interval, inequality is Pareto improving in the good equilibrium and Pareto-worsening in the bad one. In this interval, rich and poor alike benefit in the good equilibrium and are harmed in the bad one from increased inequality. Spillovers play an outsized role in this.

If θ continues to rise and exceeds $\widetilde{\theta}$, then welfare may be reduced in the good equilibrium as well. This means that, while the welfare-maximizing level of inequality is the lowest possible in the bad equilibrium, there is a positive, welfare-maximizing level of inequality in the good equilibrium.

6 Comparative Statics of CCF in a Parametric Model

This section illustrates CCF in a parametric model. Assume $(u, g, \mathcal{R}, \beta)$ satisfies

(B1) The payoff function u satisfies

$$u(c) \;=\; (1-\delta)\log c \;+\; \delta c, \tag{16}$$

where $\delta > 0$.

(B2) Return function $\mathcal{R}$ is the affine function

$$\mathcal{R}(X) \;=\; \alpha X + \gamma. \tag{17}$$

(B3) Density g is represented by a lognormal distribution over wealth: $w \sim \mathcal{LN}\left(\mu, \sigma^2(\theta)\right)$ where $\sigma^2(\theta) = \left(e^{\theta} - 1\right)\mu^2$.

The parametric form (16) in (B1) satisfies both increasing and convex inequality tolerance as long as $\delta > 0$. Inserting a linear consumption term with weight δ has the effect of perturbing the model away from log (homothetic) payoffs that have no effect on CCF. Assumption (B2) is just for simplicity. Under (B3), the variance is increasing in θ. The specification is tailored so that increases in θ leave the mean unchanged.

The consumption-savings game satisfying (B1) and (B3) admits closed form solution. The welfare differences for each variance can be computed.

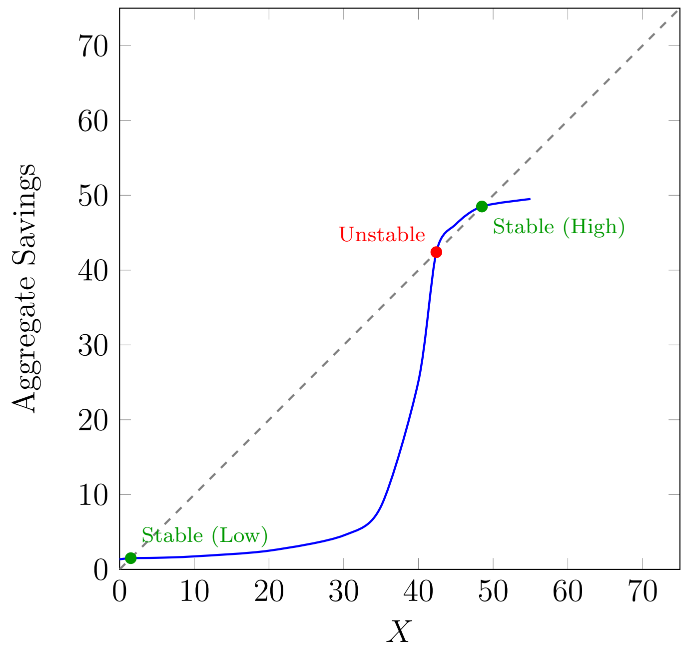

Figure 3: Multiple equilibria with Low and High stable equilibria and an unstable equilibrium.

Theorem 4. *Consider a game* $(u, g, \mathcal{R}, \beta)$ *satisfying (B1)-(B3). There exists* $b > 0$ *such that*

(i) if $\mu^2 + \sigma^2 \leq b$ *then there is a stable, neutral equilibrium* $\widehat{X} \equiv \frac{\beta}{1+\beta}\mathbb{E}[w]$ *with* $\mathcal{R}(\widehat{X}) = \frac{1}{\beta}$.

(ii) If $\mu^2 + \sigma^2 > b$ *then* $\widehat{X}$ *is an unstable Nash equilibrium and* $(u, g, \mathcal{R}, \beta)$ *admits a coordination problem, and two stable, Pareto ranked equilibria* $\underline{X}^*, \overline{X}^*$ *with* $\underline{X}^* < \widehat{X} < \overline{X}^*$ *emerge.*

The bound b is explicitly computed in the proof. It implies a bound on the variance of the wealth distribution below which the neutral equilibrium is stable. On the other hand, a large variance in wealth generates a large variance in savings behavior. In that case, self-fulfilling expectations about the size of the spillover magnifies the influence of either the low-end or high-end savers depending whether the spillover is large or small, as described after Lemma 3.

Figure 3 displays the graph of a computed example when $\mu = 50$ and $\sigma(\theta) = 8.6$. These values satisfy the assumptions of Theorem 4 generating a coordination problem.[13] The Low Stable Point is $X \approx 1.5$. At the stable Low fixed point, the returns are poor, so

[13]See the Appendix for details on generating the Figure.

agents save very little. The unstable fixed point is $X \approx 42.4$. As it crosses the threshold, savings surge and overtake investment. At the stable High fixed point $X \approx 48.5$, savings eventually plateau at the mean wealth of 50, and the 45-degree line catches back up.

7 Extended Discussion

This paper establishes conditions in which the cost of coordination failure increases with the level of wealth inequality for society. Under our conditions, inequality always reduces welfare in the bad equilibrium but can raise or lower welfare in the good equilibrium.

The assumptions we utilize are neither artificial nor restrictive. They hold, in particular, for the rich and flexible class of mixed CRRA utility functions. To the best of our knowledge, there are no related results on CCF in the literature. We can only speculate on the reasons, but one possible explanation is that the HARA family popular in the literature is an exact borderline case for our assumptions. Under HARA, the equilibria are invariant to inequality.

A key implication of our assumptions is that the savings function is convex/concave under high/low returns. This property of savings behavior is in line with empirical regularities documented by the household-finance and development literatures. On the one hand, recent evidence shows that effective savings rates increase with wealth, which corresponds to accelerating steepness of the savings function. For example, Fagereng et al. (2020) document that wealthier households have access to superior investment opportunities, hence greater incentives to save. Consequently, wealth accumulation mainly takes place at the top of the distribution. Likewise, Fagereng et al. (2019) show that once capital gains are incorporated, effective saving rates rise sharply with wealth because wealthy households benefit disproportionately more from asset appreciation.

On the other hand, literature on precautionary savings in developing economies emphasizes that in low-return and high-risk environments households rather save for self-insurance rather than for long-run wealth accumulation. For instance, Deaton (1989, 1992) shows household saving decisions in West African economies in 1980s were primarily driven by short-run consumption-smoothing motives, resulting in little variation of saving rates with wealth. Similarly, Chamon and Prasad (2010) document a substantial rise in savings rates of middle- and lower-income urban households in China in 1990-2005, made in response to reductions in public provision of housing, healthcare, and education. These empirical findings suggest that the shape of the savings function may depend on how good or bad the economic environment is, as in the present model.

The finding that inequality generates and increases the cost of coordination failure in a broad setting suggests critical linkages between inequality and poverty traps. Above all, poverty traps will be relatively worse in more unequal societies. At the same time, the worse the bad equilibrium, the greater the incentive for state involvement to nudge investors toward the superior one. Whether this can be accomplished may depend on state capacity which itself may be affected by inequality. We think an examination of policy options is critical. A desirable goal is to find the policy tool that renders coordination failure (and, consequently, research on the subject) obsolete.

8 Proofs of the Results

To simplify notation in this appendix, subscripts (e.g., $s_w(w, R)$) will be used to denote partial derivatives for those functions that contain two or more arguments. For functions with one argument, standard use of primes (e.g., u', u'', etc.) or differential notation is used.

8.1 Proof of Lemma 1

To simplify notation in the proof, drop X from the notation and consider a return R (a scalar). Evaluating the first order condition (3) at $s(w, R)$ and totally differentiating it with respect to w yields.

$$\frac{1 - s_w(w,R)}{s_w(w,R)} = \frac{\beta R^2 u''\left(Rs(w,R)\right)}{u''\left(w - s(w,R)\right)} \tag{18}$$

Notice that the left-hand side of (18) is decreasing in w if and only if s is convex and is increasing in w if s is concave.

We first prove statement (1) T_u strictly convex implies statement (2) s is convex iff $R > 1/\beta$. To prove the result, it suffices to show the right-hand side is decreasing under these conditions.

Differentiating the right-hand side of (18) with respect to w, dropping the arguments of function s, we require:

$$\frac{u^{''}(w-s)u^{'''}(Rs)Rs_w - u^{'''}(w-s)(1-s_w)u^{''}(Rs)}{(u''(w-s))^2} < 0 \tag{19}$$

(19) holds whenever

$$u^{''}(w-s)u^{'''}(Rs)Rs_w < u^{'''}(w-s)(1-s_w)u^{''}(Rs) \tag{20}$$

Substituting the expressions for s_w and $(1 - s_w)$ from (18) into (20), the required inequality (20) becomes

$$u'''\left(Rs\right)\left(u''\left(w-s\right)\right)^2 > (\geq)\ \beta R u'''\left(w-s\right)\left(u''\left(Rs\right)\right)^2 \tag{21}$$

From the first order condition (3),

$$\beta R \;=\; \frac{u'\left(w-s\right)}{u'\left(Rs\right)}. \tag{22}$$

and so substituting (22) into (21) yields

$$T_u'(Rs) \;=\; \frac{u'\left(Rs\right)u'''\left(Rs\right)}{\left(u''\left(Rs\right)\right)^2} \;>\; (\geq)\; \frac{u'\left(w-s\right)u'''\left(w-s\right)}{\left(u''\left(w-s\right)\right)^2} \;=\; T_u'(w-s) \tag{23}$$

By convex inequality tolerance this inequality holds (weakly) when

$$Rs \;>(\geq)\; w \;-\; s \tag{24}$$

which, using (22), holds precisely when $R > (\geq) 1/\beta$.

The argument for proving s is concave whenever $R < 1/\beta$ is analogous with all inequalities reversed. We omit the details.

Finally, the argument for proving statement (2) s is convex iff $R > 1/\beta$ implies statement (1) T_u is strictly convex, use our earlier observation that Equations (24) holds precisely when $R > (\geq) 1/\beta$. Then use the strict version of (24) to imply (23). This yields the convexity of T_u. ■

8.2 The Remark

Here we prove: Inequality tolerance $T_u(c)$ is strictly increasing and strictly convex if

$$u(c) = \sum_{j=1}^{J} \delta_i \frac{c^{1-\gamma_j} - 1}{1-\gamma_j}, \tag{25}$$

where $\gamma_j, \delta_j \in (0,1)$ for all j, with $\gamma_k \neq \gamma_j$ for at least one k and j pair, and $\sum_{j=1}^{J} \delta_j = 1$.

To show this, we first establish:

Step 1. Let Z be a random variable which satisfies $0 \leq Z \leq 1$ with probability one. Then, the following inequality holds:

$$2\frac{\mathbb{E}\left[Z(1+Z)\right]}{\mathbb{E}\left[Z\right]} > \mathbb{E}\left[Z\right] + \frac{\mathbb{E}\left[Z(1+Z)(2+Z)\right]}{\mathbb{E}\left[Z(1+Z)\right]}. \tag{26}$$

To prove this step, use the standard notation for the mean, $\mu = \mathbb{E}\left[Z\right]$, and variance $\sigma^2 = \mathbb{E}\left[(Z-\mu)^2\right]$. Inequality (26) can be equivalently restated as follows:

$$\frac{\mathbb{E}\left[Z^3\right] - \mu^3}{\sigma^2} < 1 + 3\mu + 2\frac{\sigma^2}{\mu}.$$

Also, the following identity holds:

$$\mathbb{E}\left[Z^3\right] - \mu^3 = \mathbb{E}\left[(Z-\mu)^3\right] + 3\mu\sigma^2.$$

Hence, one can further recast (26) as follows:

$$\frac{\mathbb{E}\left[(Z-\mu)^3\right]}{\sigma^2} < 1 + 2\frac{\sigma^2}{\mu}. \tag{27}$$

The LHS/RHS of (27) is less/greater that one, because $\sigma^2 > 0$ and

$$\mathbb{E}\left[(Z-\mu)^3\right] < \underbrace{\mathbb{E}\left[|Z-\mu|^3\right] < \mathbb{E}\left[|Z-\mu|^2\right]}_{\text{beause } |Z-\mu|<1 \text{ with probability one}} = \sigma^2.$$

Hence, (27) always holds, and so does (26), completing the proof of Step 1.

Step 2. Let

$$\rho_u(c) := -\frac{cu''(c)}{u'(c)}; \qquad \pi_u(c) := -\frac{cu'''(c)}{u''(c)}.$$

From (5),

$$\rho_u(c) = \frac{\sum_{i=1}^n \gamma_i \delta_i c^{-\gamma_i}}{\sum_{i=1}^n \delta_i c^{-\gamma_i}}; \tag{28}$$

$$\pi_u(c) = \frac{\sum_{i=1}^n \gamma_i\left(1+\gamma_i\right)\delta_i c^{-\gamma_i}}{\sum_{i=1}^n \gamma_i \delta_i c^{-\gamma_i}} = 1 + \frac{\sum_{i=1}^n \gamma_i^2 \delta_i c^{-\gamma_i}}{\sum_{i=1}^n \gamma_i \delta_i c^{-\gamma_i}}; \textit{and} \tag{29}$$

$$-\frac{cu''''(c)}{u'''(c)} = \frac{\sum_{i=1}^n \gamma_i\left(1+\gamma_i\right)\left(2+\gamma_i\right)\delta_i c^{-\gamma_i}}{\sum_{i=1}^n \gamma_i\left(1+\gamma_i\right)\delta_i c^{-\gamma_i}}. \tag{30}$$

Let us first show that $T_u(c)$ is increasing. We know that

$$T_u'(c) = \frac{\pi_u(c)}{\rho_u(c)} - 1.$$

'From $\gamma_i \in (0,1)$ and (28) — (29),

$$0 < \rho_u(c) < 1 < \pi_u(c) \implies T_u'(c) > 0.$$

This proves that $T_u(c)$ is increasing.

Let us now prove that $T_u(c)$ is convex, that is, $T_u''(c) > 0$. We exploit the following equivalence:

$$T_u''(c) \gtreqless 0 \Longleftrightarrow \frac{d\log\left(1+T_u'(c)\right)}{d\log(c)} \gtreqless 0 \Longleftrightarrow \frac{cu''''(c)}{u'''(c)} - \rho_u(c) + 2\pi_u(c) \gtreqless 0$$

Thus, we need to prove the following inequality:

$$2\pi_u(c) > \rho_u(c) - \frac{cu''''(c)}{u'''(c)}. \tag{31}$$

Define a random variable Γ as follows:

$$\mathbb{P}\left[\Gamma = \gamma_i | \boldsymbol{\delta}, c\right] = \frac{\sum_{j=1}^{n} \delta_i c^{-\gamma_i}}{\sum_{j=1}^{n} \delta_j c^{-\gamma_j}} \tag{32}$$

Using this notation, one can readily verify that

$$\rho_u(c) = \mathbb{E}\left[\Gamma | \boldsymbol{\delta}, c\right],$$

$$\pi_u(c) = \frac{\mathbb{E}\left[\Gamma\left(\Gamma+1\right)|\boldsymbol{\delta}, c\right]}{\mathbb{E}\left[\Gamma|\boldsymbol{\delta}, c\right]},$$

$$-\frac{cu^{(4)}(c)}{u'''(c)} = \frac{\mathbb{E}\left[\Gamma\left(\Gamma+1\right)\left(\Gamma+2\right)|\boldsymbol{\delta}, c\right]}{\mathbb{E}\left[\Gamma\left(\Gamma+1\right)|\boldsymbol{\delta}, c\right]},$$

and one can restate (31) as follows:

$$2\frac{\mathbb{E}\left[\Gamma\left(\Gamma+1\right)|\boldsymbol{\delta}, c\right]}{\mathbb{E}\left[\Gamma|\boldsymbol{\delta}, c\right]} > \mathbb{E}\left[\Gamma|\boldsymbol{\delta}, c\right] + \frac{\mathbb{E}\left[\Gamma\left(\Gamma+1\right)\left(\Gamma+2\right)|\boldsymbol{\delta}, c\right]}{\mathbb{E}\left[\Gamma\left(\Gamma+1\right)|\boldsymbol{\delta}, c\right]},$$

which holds true by Step 1. This completes the proof of the Remark. ■

8.3 Deriving the Decomposition in the CCF

We write:

$$\Delta V(\theta) \;=\; \int_{\underline{R}^*(\theta)}^{\overline{R}^*(\theta)} V_R(R,\theta)dR = \int_{\underline{R}^*(\theta)}^{\overline{R}^*(\theta)} \int_0^{\infty} v_R(w,R) g\left(w|\theta\right) dw\, dR. \tag{33}$$

The CCF decomposition in Equation (13) can therefore be derive by differentiating $\Delta V(\theta)$ with respect to θ.

8.4 Proof of Lemma 2

It suffices to show that the Arrow-Pratt measure of risk aversion for $v(w, \overline{R}^*)$ is lower than that of $v(w, \underline{R}^*)$. To simplify notation, let $\overline{s}(w) = s(w, \overline{R}^*)$ and $\underline{s}(w) = s(w, \underline{R}^*)$

For a given R and savings s, the first derivative is

$$v_w(w, R) = u'(w - s)(1 - s_w) + \beta R s_w u'(Rs)$$

which, by the Envelope Theorem, reduces to

$$v_w(w, R) = u'(w - s)$$

The second derivative is

$$v_{ww}(w, R) = u''(w - s)(1 - s_w)$$

The Arrow Pratt measure are $-\frac{u'(w-\bar{s}(w))}{u''(w-\bar{s}(w))(1-\bar{s}_w)}$ and $-\frac{u'(w-\underline{s}(w))}{u''(w-\underline{s}(w))(1-\underline{s}_w)}$, respectively. We therefore need to show

$$-\frac{u'(w - \bar{s}(w))}{u''(w - \bar{s}(w))(1 - \bar{s}_w)} < -\frac{u'(w - \underline{s}(w))}{u''(w - \underline{s}(w))(1 - \underline{s}_w)}$$

By decreasing absolute risk aversion of u, it suffices to show $\bar{s}_w(w) > \underline{s}_w(w)$. This holds by the fact that $\bar{s}(w) > \underline{s}(w)$ at every w and convexity of $\bar{s}$ and concavity of $\underline{s}$ in Lemma 1. Hence, we conclude $v(w, \bar{R}^*)$ has a higher Arrow-Pratt measure of absolute risk aversion than $v(w, \underline{R}^*)$. ■

8.5 Proof of Lemma 3

It suffices to show $\frac{d\overline{X}^*(\theta)}{d\theta} > 0$ and $\frac{d\overline{X}^*(\theta)}{d\theta} < 0$.

By definition,

$$\overline{X}^*(\theta) = \int s(w, R(\overline{X}^*(\theta))g(w|\theta)dw$$

and

$$\underline{X}^*(\theta) = \int s(w, R(\underline{X}^*(\theta))g(w|\theta)dw$$

Taking the high investment equilibrium, we have

$$\frac{d\overline{X}^*(\theta)}{d\theta} = \int s(w, R(\overline{X}^*(\theta))g_\theta(w|\theta)dw + \int s_R(w, R(\overline{X}^*(\theta))R_X\overline{X}^*_\theta(\theta)g(w|\theta)dw$$

or, re-arranging terms

$$\frac{d\overline{X}^*(\theta)}{d\theta} = \frac{\int s(w, R(\overline{X}^*(\theta))g_\theta(w|\theta)dw}{1 - \int s_R(w, R(\overline{X}^*(\theta))R_X g(w|\theta)dw}$$

By stability of equilibria, the denominator is positive. The numerator is positive since, by Lemma 1 and $\overline{R}^*(\theta) > 1/\beta$, the savings function s is convex and an increase in θ is a mean-preserving spread. A similar argument shows that $\frac{d\underline{X}^*(\theta)}{d\theta} < 0$, relying on the fact that s is concave when $\underline{R}^*(\theta) < 1/\beta$. ■

8.6 Proof of Lemma 4

Let us first evaluate the individual savings function $s(w, R)$ and aggregate savings function $S(R|\ \theta)$ at $R = 1/\beta$. When $R = \frac{1}{\beta}$, the individual's FOC $u'(w - x) = \beta R u'(Rx)$ becomes:

$$u'(w-x) = u'\left(\frac{1}{\beta}x\right) \implies w - x = \frac{1}{\beta}x \implies s\left(w, \frac{1}{\beta}\right) = \frac{\beta}{1+\beta}w.$$

Hence, the aggregate savings are given by

$$S\left(\frac{1}{\beta}, \theta\right) := \mathbb{E}\left[s\left(w, \frac{1}{\beta}\right), \theta\right] = \frac{\beta}{1+\beta}\mathbb{E}[w]. \tag{34}$$

Using (34), one can readily verify fixed point condition

$$S\left(\mathcal{R}(\frac{\beta}{1+\beta}\mathbb{E}[w])\Big|\ \theta\right) = S\left(\frac{1}{\beta}\Big|\ \theta\right) = \frac{\beta}{1+\beta}\mathbb{E}[w])$$

and so $X^* = \frac{\beta}{1+\beta}\mathbb{E}[w]$ is a neutral equilibrium when $\mathcal{R}(X^*) = \frac{1}{\beta}$. ■

8.7 Proof of Theorem 2

To simplify notation in what follows define

$$\rho_u(c) \equiv \frac{c}{T_u(c)} \equiv -\frac{u''(c)\,c}{u'(c)} \tag{35}$$

and

$$\alpha \equiv \mathcal{R}'\left(\frac{\beta}{1+\beta}\mathbb{E}[w]\right). \tag{36}$$

Equation (35) introduces the Arrow-Pratt measure of relative risk aversion as the inverse of inequality tolerance per unit consumption. In Equation (36), α denotes the marginal return in the neutral equilibrium.

We now prove the Theorem 2 via a series of Lemmas.

Lemma 5. *Suppose the consumption-savings game* (u, g, R, β) *admits a* neutral equilibrium *at every* θ*. Then the neutral equilibrium is stable if and only if*

$$\frac{\alpha\beta^2}{1+\beta}\mathbb{E}\left[\frac{w}{1+\beta}\frac{1-\rho_u\left(\frac{w}{1+\beta}\right)}{\rho_u\left(\frac{w}{1+\beta}\right)}\middle|\theta\right] < 1, \tag{37}$$

Proof of Lemma 5. The key property of α defined in (36) is that it is independent of θ.

The stability condition for the neutral equilibrium is given by

$$\alpha S_R\left(\frac{1}{\beta}\middle|\theta\right) < 1, \tag{38}$$

where α, defined by (36), is independent of θ. It remains to show that $\alpha S_R\left(\frac{1}{\beta}\middle|\,\theta\right)$ equals the left-hand side expression in (37).

Let us first compute $s_R(w, R)$ at $R = \frac{1}{\beta}$. By differentiating the individual's first order condition with respect to R, we get:

$$s_R(w,R) = -\beta\frac{u'(Rs(w,R)) + Ru''(Rs(w,R))\,s(w,R)}{u''(w-s(w,R)) + \beta R^2 u''(Rs(w,R))} \tag{39}$$

Plugging $R = \frac{1}{\beta}$ and $s\left(w, \frac{1}{\beta}\right) = \frac{\beta}{1+\beta}w$ into the Right-hand side of (39) and rearranging terms, we obtain:

$$s_R\left(w, \frac{1}{\beta}\right) = -\frac{\beta^2}{1+\beta}\frac{u'\left(\frac{w}{1+\beta}\right) + u''\left(\frac{w}{1+\beta}\right)\frac{w}{1+\beta}}{u''\left(\frac{w}{1+\beta}\right)} = \frac{\beta^2}{1+\beta}\frac{w}{1+\beta}\frac{1-\rho_u\left(\frac{w}{1+\beta}\right)}{\rho_u\left(\frac{w}{1+\beta}\right)}. \tag{40}$$

Applying the expectation operator $\mathbb{E}[\cdot|\theta]$ to both sides of (40), we obtain the left-hand side expression in (37) which implies stability, i.e., the inequality in (37). ■

Next, we delineate bounds on the θ space which determines stability of the neutral equilibrium.

Lemma 6. *Suppose* $T_u(c) > c$ *for all* c*. Then* $S_R\left(\frac{1}{\beta}, \theta\right)$ *increases without bound as* $\theta \to \infty$*.*

Remark By definition, $T_u(c) > c \iff \rho_u(c) < 1$. That is, risk-tolerance increasing at rate greater than c is equivalent to relative risk aversion being inelastic. The point of Lemma 6 is to show that the the stability condition (37) holds for low values of θ and fails to hold for high values of θ. For this, it suffices to show the left-hand side of (37) is increasing without bound with respect to θ.

Proof of Lemma 6. From Lemma 5, we need to prove is that $S_R\left(\frac{1}{\beta}, \theta\right)$ increases without bound as $\theta \to \infty$ under convex T_u.

Now standard second order stochastic properties of the mean-preserving spread, $S_R\left(\frac{1}{\beta}, \theta\right)$ increases in θ iff $s_R\left(w, \frac{1}{\beta}\right)$ is convex with respect to w. From (40), convexity of $s_R\left(w, \frac{1}{\beta}\right)$ with respect to w is equivalent to

$$\frac{d^2}{dc^2}\left[T_u(c) - c\right] \equiv \frac{d^2}{dc^2}\left[\frac{u'(c)}{-u''(c)} - c\right] > 0.$$

It remains to observe the following identity holds:

$$\frac{d^2}{dc^2}\left[\frac{u'(c)}{-u''(c)} - c\right] = \frac{d}{dc}\left[\frac{u'(c)\,u'''(c)}{u''(c)^2}\right].$$

Next, observe that, by definition, $T_u(c) > c$ iff $\rho(c) < 1$. Since, by the previous Lemma, The proof the previous lemma showed

$$S_R\left(\frac{1}{\beta}, \theta\right) = \frac{\beta^2}{1+\beta}\mathbb{E}\left[\frac{w}{1+\beta}\frac{1-\rho_u\left(\frac{w}{1+\beta}\right)}{\rho_u\left(\frac{w}{1+\beta}\right)}\middle|\,\theta\right] \tag{41}$$

Then $S_R\left(\frac{1}{\beta}\middle|\,\theta\right) > 0$ for all θ.

Recall from Equation (40),

$$S_R\left(\frac{1}{\beta}\middle|\,\theta\right) = \int s_R\left(w, \frac{1}{\beta}\right) g(w|\theta)dw = \frac{\beta^2}{1+\beta}\mathbb{E}\left[\frac{w}{1+\beta}\frac{1-\rho_u\left(\frac{w}{1+\beta}\right)}{\rho_u\left(\frac{w}{1+\beta}\right)}\middle|\,\theta\right]$$

which we can write as

$$S_R\left(\frac{1}{\beta}\middle|\,\theta\right) = \frac{\beta^2}{1+\beta}\mathbb{E}\left[\frac{1}{1+\beta}\frac{1-\rho_u\left(\frac{w}{1+\beta}\right)}{\frac{\rho_u\left(\frac{w}{1+\beta}\right)}{w}}\middle|\,\theta\right].$$

By our assumption that $T_u(c) > c$, and hence, $\rho_u(c) < 1$, it follows that $\rho_u(w)/w$ is $\mathcal{O}(\frac{1}{w})$ hence diverges in w faster than an $\mathcal{O}(w^2)$ function. Thus there exists θ° such that

$$S_R\left(\frac{1}{\beta}\middle|\,\theta\right) > \mathbb{E}\left[w^2\middle|\,\theta\right] \quad \forall\ \theta > \theta^\circ$$

But since $\mathbb{E}\left[w^2\middle|\,\theta\right] \to \infty$ as $\theta \to \infty$, it follows that $S_R\left(\frac{1}{\beta}\middle|\,\theta\right) \to \infty$ as well. This completes the proof. ■

Lemma 7. *Suppose the consumption-savings game (u, g, R, β) admits an unstable neutral equilibrium. Suppose for all c,*

(i) $T'_u(c) > 0$,

(ii) $T''_u(c) > 0$, *and*

(iii) $T_u(c) > c$

Then, there always exist at least two stable equilibria, $\underline{X}^*(\theta)$ and $\overline{X}^*(\theta)$, such that $\underline{X}^*(\theta) < \frac{\beta}{1+\beta}\mathbb{E}\left[w\right] < \overline{X}^*(\theta)$.

Proof of Lemma 7. The instability of the neutral equilibrium means $S_R\left(\mathcal{R}(X)|\theta\right)\mathcal{R}'(X) > 1$ at $X = \frac{\beta}{1+\beta}\mathbb{E}\left[w\right]$. Hence, there exists $\epsilon > 0$, such that $S\left(\mathcal{R}(X)|\theta\right) < X$ for $X \in \left(\frac{\beta}{1+\beta}\mathbb{E}\left[w\right] - \epsilon, \frac{\beta}{1+\beta}\mathbb{E}\left[w\right]\right)$. At the same time, $\mathcal{R}(0) > 0$ implies $S\left(\mathcal{R}(X)|\theta\right) > X$ for X sufficiently close to zero. Hence, the fixed point condition $X = S\left(\mathcal{R}(X)|\theta\right)$ has at least one solution $\underline{X}^*(\theta) \in \left(0, \frac{\beta}{1+\beta}\mathbb{E}\left[w\right]\right)$, such that $S_R\left(\mathcal{R}(X)|\theta\right)\mathcal{R}'(X) < 1$ at $X = \underline{X}^*(\theta)$.

To prove the existence of $\overline{X}^*(\theta)$, it suffices to show that $S\left(\mathcal{R}(X)|\theta\right) < X$ when X is sufficiently large. This, in turn, is the case when $S_R\left(\mathcal{R}(X)|\theta\right)\mathcal{R}'(X) < 1$ for sufficiently large values of X. Two cases may arise.

Case 1: $\mathcal{R}(\infty) < \infty$. In this case, $\mathcal{R}'(\infty) = 0$, hence,

$$\lim_{X\to\infty}\left[S_R\left(\mathcal{R}(X)|\theta\right)\mathcal{R}'(X)\right] = 0 < 1.$$

Case 2: $\mathcal{R}(\infty) = \infty$. In this case, we have:

$$\lim_{X\to\infty}\left[S_R\left(\mathcal{R}(X)|\theta\right)\mathcal{R}'(X)\right] = \lim_{R\to\infty}\left[RS_R\left(R|\theta\right)\right]\underbrace{\lim_{X\to\infty}\left[\frac{\mathcal{R}'(X)}{\mathcal{R}(X)}\right]}_{=0}. \tag{42}$$

The second term in the RHS of (42) is zero due to the initial assumptions on $\mathcal{R}$, in this case, the assumption of sub-exponential growth:

$$\lim_{X\to\infty}\frac{\mathcal{R}'(X)}{\mathcal{R}(X)}=0; \tag{43}$$

Thus, we need to show that

$$\lim_{R\to\infty} RS_R\left(R|\theta\right)<\infty, \tag{44}$$

which will imply that

$$\lim_{X\to\infty}\left[S_R\left(\mathcal{R}(X)|\theta\right)\mathcal{R}'(X)\right]=0,$$

hence $S_R\left(\mathcal{R}(X)|\theta\right)\mathcal{R}'(X)<1$ for sufficiently large values of X.

Let us restate the expression (39) for $s_R\left(w,R\right)$ as follows (we suppress the arguments of the savings function $s(w,R)$ in the RHS):

$$s_R\left(w,R\right)=\beta u'\left(Rs\right)\frac{1-\rho_u\left(Rs\right)}{-u''\left(w-s\right)-\beta R^2u''\left(Rs\right)}. \tag{45}$$

From the individual's FOC, we get:

$$\beta R=\frac{u'\left(w-s\right)}{u'\left(Rs\right)}\quad\text{and}\quad\beta u'\left(Rs\right)=\frac{u'\left(w-s\right)}{R}. \tag{46}$$

Plugging, (46) into (45), and using $\rho_u\left(\cdot\right)<1$ and $T_u\left(\cdot\right)>0$, we get:

$$s_R\left(w,R\right)=\frac{1}{R}\frac{1-\rho_u\left(Rs\right)}{\frac{1}{T_u(w-s)}+R\frac{1}{T_u(Rs)}}<\frac{1}{R}\frac{1}{\frac{1}{T_u(w-s)}+R\frac{1}{T_u(Rs)}}<\frac{1}{R}\frac{1}{\frac{1}{T_u(w-s)}},$$

hence,

$$s_R\left(w,R\right)<\frac{T_u\left(w-s\right)}{R}. \tag{47}$$

Define the following improper integral:

$$\mathcal{A}:=\int_1^\infty\frac{d\xi}{T_u\left(\xi\right)}. \tag{48}$$

Two sub-cases may arise.

Sub-case 2(a): The integral in (48) converges, that is, $\mathcal{A}<\infty$.

In this case, $u'(\infty)>0$. This follows from the identity

$$u'(c) = u'(1)\exp\left\{-\int_1^c \frac{d\xi}{T_u(\xi)}\right\} \Longrightarrow u'(\infty) = u'(1)e^{-\mathcal{A}} > 0.$$

Denote $\delta := u'(\infty)$ (this notation is in line with our leading example). Then, the FOC implies

$$u'(w-s) = \beta R u'(RS) > \beta R u'(\infty) = \delta\beta R,$$

hence,

$$w - s < u'^{-1}(\delta\beta R) \underbrace{\Longrightarrow T_u(w-s) < \left(T_u \circ u'^{-1}\right)(\delta\beta R)}_{\text{from } T_u'(\cdot)>0}. \tag{49}$$

From (47) and (49), we get:

$$s_R(w,R) < \frac{1}{R}\left(T_u \circ u'^{-1}\right)(\delta\beta R). \tag{50}$$

As (50) holds for all w, we get:

$$RS_R(R|\theta) = R\mathbb{E}\left[s_R(w,R)|\theta\right] < \left(T_u \circ u'^{-1}\right)(\delta\beta R). \tag{51}$$

Taking the limit under $R \to \infty$ on both sides of (51), we get:

$$\lim_{R\to\infty}\left[RS_R(R|\theta)\right] \leq \lim_{R\to\infty}\left[T_u\left(u'^{-1}(\delta\beta R)\right)\right] < \infty,$$

because $(T_u \circ u'^{-1})'(\cdot) < 0 < (T_u \circ u'^{-1})(\cdot)$. This proves (44) for sub-case 2(a).

Sub-case 2(b): The integral in (48) diverges, that is, $\mathcal{A} = \infty$.

In this case, $T_u(c)$ increases slower than c^{-2}, because otherwise the integral in (48) would converge. Thus, we have:

$$T_u(w) = \circ\left(w^2\right) \text{ when } w \to \infty. \tag{52}$$

Using (52) and the finite variance assumption, we get

$$\mathbb{E}\left[w^2\middle|\theta\right] < \infty \Longrightarrow \mathbb{E}\left[T_u(w)|\theta\right] < \infty.$$

Combining (47) with the assumption of increasing inequality tolerance, $T_u'(\cdot) > 0$, we get:

$$s_R(w,R) < \frac{1}{R}T_u(w). \tag{53}$$

From (53),

$$RS(R|\theta) := \mathbb{E}\left[Rs_R(w,R)|\theta\right] < \mathbb{E}\left[T_u(w)|\theta\right] < \infty. \tag{54}$$

Taking the limit under $R \to \infty$ in (54), we get:

$$\lim_{R\to\infty}\left[RS_R(R|\theta)\right] \le \mathbb{E}\left[T_u(w)|\theta\right] < \infty.$$

This proves (44) for sub-case 2(b) and completes the proof. ■

Remainder of Proof of Theorem 2. From Lemma 4, the neutral equilibrium exists under (15) and is stable by Lemma 5 if (37) holds. Lemma 6 shows $\alpha S_R\left(\frac{1}{\beta}\middle|\theta\right) \to \infty$ as $\theta \to \infty$. Since $\alpha S_R\left(\frac{1}{\beta}\middle|\theta\right) < 1$ at $\theta = 0$, it follows from the Intermediate Value Theorem that there exists $\widehat{\theta}$ such that $\alpha S_R\left(\frac{1}{\beta}\middle|\theta\right) \le 1$ for all $\theta \le \widehat{\theta}$, and $\alpha S_R\left(\frac{1}{\beta}\middle|\theta\right) > 1$ for all $\theta > \widehat{\theta}$. Hence when $\theta \le \widehat{\theta}$, the neutral equilibrium is unstable, and when $\theta > \widehat{\theta}$, the neutral equilibrium becomes unstable. Now apply Lemma 7 to show that two other, stable solutions emerge satisfying the criteria (a)-(d) when $\theta > \widehat{\theta}$. ■

8.8 Proof of Theorem 3

The proof essentially amounts to showing that

$$\lim_{\theta\to\widehat{\theta}} \overline{V}^*_\theta(\theta) = +\infty.$$

The bifurcation which occurs when $\theta \to \widehat{\theta}$ is the pitchfork bifurcation. Under this type of bifurcation, the limiting behavior of $\overline{X}^*(\theta)$ and $\underline{X}^*(\theta)$ under $\theta \to \widehat{\theta}$ is as follows:

$$\overline{X}^*(\theta) - \widehat{X} \sim K\sqrt{\theta - \widehat{\theta}},$$

$$\underline{X}^*(\theta) - \widehat{X} \sim -K\sqrt{\theta - \widehat{\theta}},$$

where $K > 0$ is a constant. Hence,

$$\lim_{\theta\searrow\widehat{\theta}} \overline{X}^*_\theta(\theta) = +\infty, \qquad \lim_{\theta\searrow\widehat{\theta}} \underline{X}^*_\theta(\theta) = -\infty;$$

and, since $\overline{R}^*_\theta(\theta) = \mathcal{R}'\left(\overline{X}^*_\theta(\theta)\right)$

$$\lim_{\theta \searrow \widehat{\theta}} \overline{R}^*_\theta(\theta) = +\infty, \qquad \lim_{\theta \searrow \widehat{\theta}} \underline{R}^*_\theta(\theta) = -\infty;$$

Consider the welfare differential between the good stable equilibrium and the "symmetric" equilibrium when θ is slightly above $\widehat{\theta}$:

$$V\left(\overline{R}^*(\theta)|\theta\right) - V\left(\frac{1}{\beta}|\theta\right) = \int_{1/\beta}^{\overline{R}^*(\theta)} V_R\left(R|\theta\right) dR.$$

$$V_R\left(R|\theta\right) = \mathbb{E}\left[v_R(w,R)|\theta\right]$$

$$v_R(w,R) = \beta s\left(w,R\right) u'\left(Rs\left(w,R\right)\right)$$

For our leading example of utility:

$$v_R(w,R) = \beta s\left(w,R\right)\left(\frac{1-\delta}{Rs\left(w,R\right)} + \delta\right) = \beta\left(\delta s\left(w,R\right) + \frac{1-\delta}{R}\right).$$

$$V_R\left(R|\theta\right) = \mathbb{E}\left[v_R(w,R)|\theta\right] = \beta\left(\delta S\left(R|\theta\right) + \frac{1-\delta}{R}\right)$$

$$\frac{d}{d\theta}\left[V\left(\overline{R}^*(\theta)|\theta\right) - V\left(\frac{1}{\beta}|\theta\right)\right] = V_R\left(\overline{R}^*(\theta)|\theta\right)\overline{R}^*_\theta(\theta) + \int_{1/\beta}^{\overline{R}^*(\theta)} V_{R\theta}\left(R|\theta\right) dR \qquad (55)$$

When $\theta \to \widehat{\theta}$, the first term in the RHS of (55) goes to $+\infty$, while the second term goes to zero. Hence,

$$\lim_{\theta \to \widehat{\theta}}\left[\overline{V}^*_\theta(\theta) - \left(V^*_{\text{ntr}}\right)_\theta(\theta)\right] = +\infty.$$

Because $\left(V^*_{\text{ntr}}\right)_\theta(\theta)$ is negative but finite in the vicinity of $\theta = \widehat{\theta}$, this implies

$$\lim_{\theta \to \widehat{\theta}} \overline{V}^*_\theta(\theta) = +\infty.$$

Hence, there must be a non-empty open interval $\left(\widehat{\theta}, \widetilde{\theta}\right)$ where $\overline{V}^*_\theta(\theta) > 0$. This completes the proof. ■

8.9 Closed form calculation of savings function under parametric assumptions (B1)-(B3)

Using our parametric specification in (B1)-(B3), the first order condition becomes

$$w = \omega(x, R), \qquad \text{where} \qquad \omega(x, R) := x + \frac{x}{\beta + \frac{\delta}{1-\delta}\,(\beta R - 1)\,x}. \tag{56}$$

The function $\omega(x, R)$ shows the level of wealth of an individual who will choose to invest x units of wealth facing the return on investment equal to R. The function $\omega(x, R)$ has the following properties:

1. $\omega(x, R)$ is increasing in x:

$$\omega_x(x, R) = 1 + \frac{(1-\delta)^2\beta}{\left[(1-\delta)\beta + \delta\,(\beta R - 1)\,x\right]^2} > 0; \tag{57}$$

2. $\omega(x, R)$ is decreasing in R:

$$\omega_R(x, R) = -\frac{(1-\delta)\delta\beta x^2}{\left[(1-\delta)\beta + \delta\,(\beta R - 1)\,x\right]^2} < 0; \tag{58}$$

3. $\omega(x, R)$ is concave/convex in x iff the return on investment R is above/below the threshold $1/\beta$:

$$\omega_{xx}(x, R) = -\frac{(1-\delta)^2\delta\beta\,(\beta R - 1)}{\left[(1-\delta)\beta + \delta\,(\beta R - 1)\,x\right]^3} \lesseqgtr 0 \iff R \gtreqless \frac{1}{\beta}. \tag{59}$$

Define the individual savings function as the solution $x = s(w, R)$ to (56) w.r.t. x, which is unique due to (57). The individual savings function $s(w, R)$ satisfies $0 < s(w, R) < w$ and has the following properties, which follow from properties (57) — (59) of $\omega(x, R)$:

1. $s(w, R)$ increasing in wealth:

$$s_w(w, R) = \frac{1}{\omega_x(s(w, R), R)} = \frac{\left[\beta + \frac{\delta}{1-\delta}\,(\beta R - 1)\,x\right]^2}{\left[\beta + \frac{\delta}{1-\delta}\,(\beta R - 1)\,x\right]^2 + \beta} > 0; \tag{60}$$

2. $s(w, R)$ is increasing in investment returns:

$$s_R(w, R) = -\frac{\omega_R(s(w, R), R)}{\omega_x(s(w, R), R)} = \frac{\frac{\delta}{1-\delta}\beta}{\frac{\beta}{s^2} + \left[\frac{\beta}{s} + \frac{\delta}{1-\delta}\,(\beta R - 1)\,s\right]^2} > 0; \tag{61}$$

3. $s(w,R)$ is convex/concave in wealth iff the return on investment R is above/below the threshold $1/\beta$:

$$s_{ww}(w,R) = -\frac{\omega_{xx}(s(w,R),R)}{[\omega_x(s(w,R),R)]^2} \gtreqless 0 \Longleftrightarrow R \gtreqless \frac{1}{\beta}. \tag{62}$$

The individual savings function can be expressed from (56) in closed form:

$$w = x + \frac{1}{\beta}\frac{x}{1+\frac{\delta}{1-\delta}\left(R-\frac{1}{\beta}\right)x}$$

$$w = x\left[1+\frac{1}{\beta}\frac{1}{1+\frac{\delta}{1-\delta}\left(R-\frac{1}{\beta}\right)x}\right]$$

$$\delta\left(\beta R-1\right)x^2 + \left[(1-\delta)(1+\beta) - \delta w\left(\beta R-1\right)\right]x - (1-\delta)\beta w = 0$$

This yields,

$$s(w,R) = \frac{w}{2} + \frac{\sqrt{\left[\delta\left(\beta R-1\right)w-\left(1-\delta\right)\left(1+\beta\right)\right]^2 + 4\left(1-\delta\right)\delta\left(\beta R-1\right)\beta w} - \left(1-\delta\right)\left(1+\beta\right)}{2\delta\left(\beta R-1\right)}. \tag{63}$$

8.10 Proof of Theorem 4

Under (B1)-(B3), the stability condition (37) for the neutral equilibrium $R^* = 1/\beta$ becomes

$$e^{\theta} - 1 \; < \; \frac{1+\beta}{\alpha}\frac{1}{\mu^2}\frac{1-\delta}{\delta}\left(1+\frac{1}{\beta}\right)^2 \; - \; 1 \tag{64}$$

Next, choose μ so that

$$\mu \; < \; \left(1+\frac{1}{\beta}\right)\sqrt{\frac{1+\beta}{\alpha}\frac{1-\delta}{\delta}}. \tag{65}$$

Hence b can be constructed as the sum of right-hand sides of (64) and (65).

Then, there exists a unique $\widehat{\theta} > 0$, such that $R^* = 1/\beta$ is stable if $\theta < \widehat{\theta}$:

$$\widehat{\theta} = \ln\left[\frac{1+\beta}{\alpha}\frac{1}{\mu^2}\frac{1-\delta}{\delta}\left(1+\frac{1}{\beta}\right)^2\right] > 0.$$

As θ keeps increasing, the neutral equilibrium $R^* = 1/\beta$ becomes unstable the moment θ hits the threshols $\widehat{\theta}$, and two stable Pareto-ranked equilibria, $\overline{R}^*(\theta) > \frac{1}{\beta} > \underline{R}^*(\theta)$, emerge by Lemma 7. The closed-form expression for $\widehat{\theta}$ is as follows:

$$\widehat{\theta} = \ln\left[\frac{1+\beta}{\alpha}\frac{1}{\mu^2}\frac{1-\delta}{\delta}\left(1+\frac{1}{\beta}\right)^2\right] > 0.$$

We conclude the proof. ■

8.11 Note on generating Figure 3

The Figure was generated using the Gemini AI tool. Aggregate savings derived from closed form savings function in (63) when (B1)-(B3) hold, and with $\mu = 50$ and $\theta = 0.159$. Other parameters: $\delta = 0.5$, $\beta = 0.9$, and $R(X) = .02X + .3$. The bound b is computed from these parameters using the closed form expression for b, as constructed as the sum of right-hand sides of (64) and (65).

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